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APPLICATION OF THE FINITE POINTSET METHOD IN SOLVING DUAL-PHASE LAG EQUATION

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Abstract: *This study presents the pioneering development of Finite Pointset Method (FPM) for dual-phase lag (DPL) equation in bioheat transfer. The paper explores its mathematical formulation, possible practical applications, and concludes with a numerical example comparing FPM results to analytical solutions, demonstrating the method's accuracy and versatility. To solve DPL equation, this paper proposes FPM, a meshless numerical technique that eliminates the need for structured meshes. FPM employs scattered nodes and weighted least-squares approximation, making it particularly effective for complex geometries and irregular boundaries. Its Lagrangian formulation simplifies the enforcement of boundary conditions compared to traditional methods. DPL heat conduction model is a key advancement in heat transfer analysis, addressing limitations of classical models like Fourier's law in handling rapid heat flux and non-equilibrium conditions. By incorporating time delays for both heat flux response and temperature gradient establishment, DPL offers a more accurate depiction of thermal processes in systems with thermal inertia. Its applications extend from biological heat transfer in thermal therapies to micro- and nanotechnology, advanced materials science, and aerospace engineering.*

Keywords: heat transfer, dual-phase lag, meshless methods, Finite Pointset Method, FPM

1. Introduction

Dual-Phase Lag (DPL) heat conduction model represents a significant advancement in the field of heat transfer, particularly in contexts where classical models, such as Fourier's law, fail to capture the complex dynamics of thermal processes. DPL model addresses scenarios involving rapid heat fluxes and non-equilibrium conditions, where the assumptions of instantaneous heat propagation inherent in traditional models are no longer valid. Unlike Fourier's law, which assumes an immediate relationship between the temperature gradient and heat flux, DPL model incorporates two distinct time delays: one for the response of the heat flux to a temperature gradient (the heat flux phase lag), and another for the establishment of the temperature gradient in response to external heating (the temperature gradient phase lag). These phase lags provide a more realistic representation of heat conduction in materials and systems where thermal inertia and delayed thermal responses play critical roles (Majchrzak and Turchan, 2015).

The applicability of DPL model spans various domains beyond the conventional study of heat conduction. In biological systems, it is employed to simulate heat transfer in tissues subjected to thermal therapies, accounting for the complex thermal behavior of living matter. In micro- and nanotechnology, where classical models break down due to size effects and non-local heat transport phenomena, DPL model provides a framework for accurate thermal analysis. Additionally, it finds utility in materials science, particularly in the investigation of advanced composite materials and structures with unique thermal properties, as well as in aerospace engineering for analyzing heat transfer under extreme environmental conditions.

To solve described DPL equation the Finite Point Method (FPM) here is proposed by the author of this article. FPM stands out as a genuinely meshless approach, as it does not necessitate the creation of structured or base meshes, unlike traditional techniques such as the Finite Element Method (FEM) or the Finite Difference Method (FDM), which are typically used to solve partial differential equations or to interpolate field variables (Kuhnert, 1999). Instead, FPM relies on a collection of scattered nodes distributed throughout the problem domain and its boundaries. By employing the weighted least-squares technique, it effectively constructs accurate approximations (Reséndiz-Flores and Saucedo-Zendejo, 2015). Its

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Lagrangian formulation makes it particularly well-suited for handling problems involving highly complex geometries and irregular boundaries. Additionally, its strong form simplifies the enforcement of boundary conditions compared to other numerical methods. Due to these advantages, FPM has been successfully applied to various fields, including fluid mechanics (Tiwari and Kuhnert, 2001), heat transfer (Wawreńczuk et al., 2007), linear elasticity (Saucedo-Zendejo and Reséndiz-Flores, 2020), piezoelectric phenomena (Saucedo-Zendejo et al., 2024), and biharmonic equations relevant to thin plate bending or viscous fluid flow, among others.

This paper presents, for the first time in the scientific literature, the development and application of the FPM to the DPL equation, according to the authors' knowledge. It aims to explore the mathematical formulation of FPM to the application to DPL in a case of bioheat transfer and to discuss its practical applications in this field. Paper is concluded by numerical example of calculations, where comparison with analytical solution can be seen.

2. Dual-phase lag equation

The mathematical formulation of the DPL model introduces two characteristic parameters: the heat flux phase lag and the temperature gradient phase lag. By tuning these parameters, the model can bridge the gap between classical Fourier conduction and more complex models, such as the Cattaneo-Vernotte and hyperbolic heat conduction models. As such, DPL model serves as a versatile and powerful tool for studying heat conduction across a wide range of applications and has the following form (Majchrzak and Turchan, 2015):

$$c\,\rho\left[\frac{\partial\,T(x,t)}{\partial\,t}+\tau_q\frac{\partial^2\,T(x,t)}{\partial\,t^2}\right]=\lambda\,\nabla^2\,T(x,t)+\lambda\,\tau_T\frac{\partial\,\nabla^2\,T(x,t)}{\partial\,t}+Q(x,t)+\tau_q\frac{\partial\,Q(x,t)}{\partial\,t} \tag{1}$$

where c is the specific heat, ρ is the density, λ is the thermal conductivity, T is the temperature, t is the time and $Q(x,t)$ is the source term due to metabolism and blood perfusion. In equation (1) τ_q is the relaxation time and τ_T is the thermalization time, τ_q is the phase-lag in establishing the heat flux and associated conduction through the medium and τ_T is the phase-lag in establishing the temperature gradient across the medium.

The source term $Q(x,t)$ can be written in the following form (Majchrzak and Turchan, 2015):

$$Q(x,y)=G_B c_B[T_B-T(x,y)]+Q_m \tag{2}$$

where G_B is the blood perfusion rate, c_B is the specific heat of blood, T_B is the blood temperature and Q_m is the metabolic heat source.

It should be emphasized that for $\tau_T = 0$ DPL (1) reduces to the Cattaneo–Vernotte equation, while for $\tau_q = \tau_T = 0$ it reduces to the Pennes one and this equation contains a second order time derivative and higher order mixed derivative in both time and space.

The mathematical model to be complete must be supplemented by the adequate boundary-initial conditions that are formulated to DPL (Majchrzak and Turchan, 2015).

3. The Finite Pointset Method

Let D be given domain with a particular boundary in the 2D space and suppose that the set of points x_1, x_2, …, x_n is distributed with corresponding function values $T(x_1)$, $T(x_2)$, …, $T(x_n)$. The problem is to find an approximate value of function T at some arbitrary location x. For this purpose, let us define the approximation of $T(x_j)$ using the Taylor series expansion around x (Kuhnert, 1999):

$$T_{aprox}(x_j)=T(x)+\sum_{k=1}^{2}T_k(x)dx_j^k+\frac{1}{2}\sum_{k,l=1}^{2}T_{kl}(x)dx_j^k x_j^l \tag{3}$$

The values that are not known $T(x)$, $T_k(x)$, $T_{kl}(x)$ (k = 1, 2, l = 1, 2) are obtained from a weighted least squares method achieved by minimizing the quadratic expression while considering all neighbor points (np-number of neighbor points):

$$J=\sum_{j=1}^{np}w_j(Ma-b)^2 \tag{4}$$

where

$$w_j=w(x_j,x)=\begin{cases}\exp\left(-\beta\left\|x_j-x\right\|^2/h^2\right), & \left\|x_j-x\right\|\le h\\ 0, \text{otherwise}\end{cases} \tag{5}$$

where β is a positive constant. The size of h defines a set of neighbor points around x. After some mathematical operations, the minimization of the function J results formally in:

$$a = (M^T W M)^{-1}(M^T W)b \tag{6}$$

and

$$W = \begin{pmatrix} w(x_1, x) & 0 & \cdots & 0 \\ 0 & w(x_2, x) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(x_{np}, x) \end{pmatrix} \tag{7}$$

In this point we assume that x belongs to the interior part of D. Moreover the matrix M, the unknown vector a and the vector b for a 2D case are defined as follows:

$$M = \begin{pmatrix} 1 & dx_1^1 & dx_1^2 & \frac{1}{2}(dx_1^1)^2 & dx_1^1 dx_1^2 & \frac{1}{2}(dx_1^2)^2 \\ 1 & dx_2^1 & dx_2^2 & \frac{1}{2}(dx_2^1)^2 & dx_2^1 dx_2^2 & \frac{1}{2}(dx_2^2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & dx_{np}^1 & dx_{np}^2 & \frac{1}{2}\left(dx_{np}^1\right)^2 & dx_{np}^1 dx_{np}^2 & \frac{1}{2}\left(dx_{np}^2\right)^2 \\ A_1 & 0 & 0 & A_2 & 0 & A_2 \end{pmatrix} \tag{8}$$

$$a = [T(x), T_1(x), T_2(x), T_{11}(x), T_{12}(x), T_{22}(x)]^t \tag{9}$$

$$b = \left[T^{\tau+1}(x_1), T^{\tau+1}(x_2), \dots, T^{\tau+1}\left(x_{np}\right), A_3\right]^t \tag{10}$$

where

$$A_1 = c\rho\left(\Delta t + \tau_q\right) \tag{11}$$

$$A_2 = -\lambda \Delta t (\Delta t + \tau_T) \tag{12}$$

$$A_3 = T^\tau(x) c\rho\left(\Delta t + 2\tau_q\right) - c\rho\tau_q T^{\tau-1}(x) - \lambda \Delta t \tau_T \nabla^2 T^\tau + Q^\tau(x)\Delta t^2 + \tau_q \Delta t\left(Q^{\tau+1}(x) - Q^\tau(x)\right) \tag{13}$$

FPM operates as an iterative technique where the vector a in equation (9) is recomputed over each particle till the chosen stopping criterion is satisfied.

It is worth mentioning that if point x belongs to the edge of D and satisfies the second type of boundary condition, one extra row must be added in matrix (8): $\left[0, \left(1 + \frac{\tau_T}{dt}\right) n_x, \left(1 + \frac{\tau_T}{dt}\right) n_y, 0,0,0\right]$ and one extra element in vector (10): $\frac{-q_b}{\lambda} + \frac{\tau_T}{dt}(T_1^\tau n_1 + T_2^\tau n_2)$, because we have one equation more. Time moment is denoted as τ.

4. Numerical examples

In order to verify the accuracy of FPM the following example of boundary-initial problem has been solved. First, the distribution of the temperature in 1D domain is determined by the following equation (Majchrzak and Turchan, 2015):

$$\frac{\partial T(x,t)}{\partial t} + \left(100 + \frac{1}{\pi^2}\right) \frac{\partial^2 T(x,t)}{\partial t^2} = \frac{\partial^2 T(x,t)}{\partial x^2} + \left(10^{-6} + \frac{1}{\pi^2}\right) \frac{\partial^3 T(x,t)}{\partial t \partial x^2} \tag{14}$$

This equation is supplemented by the boundary conditions: $T(0,t) = T(L,t) = 0$ and initial conditions: $T(x,0) = sin(10^4 \pi x), \frac{\partial T(x,t)}{\partial t} = -\pi^2 sin(10^4 \pi x)$ for $t = 0$.

Moreover, analytical solution of this problem is of the form:

$$T(x,\ t) = exp(-\pi^2 t) sin(10^4 \pi x) \tag{15}$$

The solution (15) is the same in a case of square domain in 2D for which the Dirichlet conditions are assumed on two opposite edges, while for the remaining two boundaries no-flux conditions are taken into account. This task has been solved under the assumptions that: geometry of the 2D area is L x L for $L = 10^{-4}$, time step dt =0.01, lattice step $dx = dy = 10^{-5}$ and $\beta = 6$.

In Fig. 1 and 2 the comparison of numerical and analytical solutions for 1D and 2D problem is shown. For the 2D problem, points representing the axis of symmetry of the area were taken to present the results. A good agreement between both solutions with the maximum relative error 0.4% for 1D and 2% for 2D is visible.

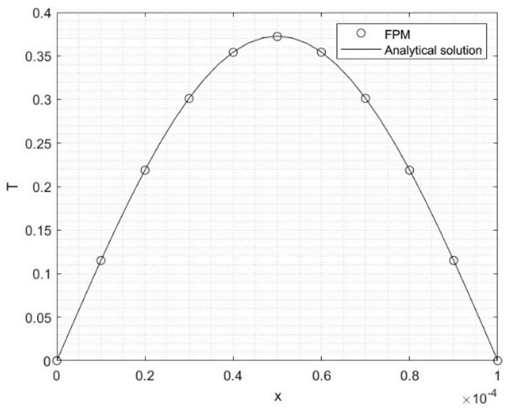

Fig. 1:Comparison in 1D domain for t = 0.1.

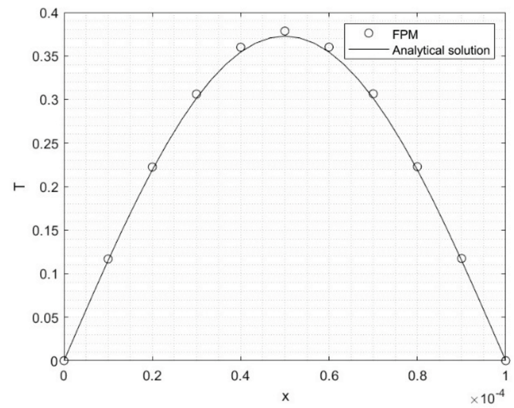

Fig. 2: Comparison in 2D domain for t= 0.1.

5. Conclusions

FPM can be effectively applied to the problems based on dual-phase lag equation. It easily accommodates time-dependent variations in temperature and can capture the evolution of temperature profiles over time, making it suitable for simulating dynamic heat transfer scenarios. Analysed numerical technique is well-suited for solving complex problems involving heat transfer in biological tissues what will be next step in author's investigations. Additional advantage of this method is very simple boundary condition way of application, even those more complicated suited for DPL models. This fact creates possibility of its future application to three-phase equation. The accuracy of FPM method is considered as very good, as could be observed in Fig. 1 and 2 where comparison with the analytical solution is presented.

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