NEW MATHEMATICAL MODEL OF CERTAIN CLASS OF CONTINUUM MECHANICS PROBLEMS

František Pochylý*, Simona Fialová*, Jaroslav Krutil*

This paper presents a variant of a mathematical model of continuum mechanics. Adaptation of the model is focused on the unsteady term. The solution is based on the assumption of the zero value of the divergence vector, which can have a different physical meaning.

Keywords: *fluid structure interaction, momentum equation, continuum mechanics, Maxwell equations*

1. Introduction

Solution of many problems of continuum mechanics is based on the method of control volumes. Formulation of the task is often complicated by the unsteady term on which in the classic formulation cannot be applied the Gauss-Ostrogradsky theorem.

The above mentioned unsteady term often complicates the problematic of the fluid structure interaction. Major complications occur even in dealing with the interactions of fields of different physical nature; for example the interaction of the fluid and electromagnetic fields. For simplification there is used the index notation in the article.

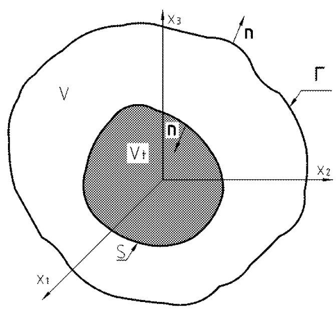

Fig.1: The body motion in the incompressible liquid

Here are two examples:

- Let's determine the strength exerted incompressible fluid moving object, as shown in Figure 1: i-th component of the force acting on the body is defined by the term [4], [5], where $\mathbf{n}$ is inner nominal vector.

$$F_i = -\int_S \sigma_{ij}\, n_j \,\mathrm{d}S \ . \tag{1}$$

* prof. Ing. F. Pochylý, CSc., Ing. S. Fialová, Ph.D., Ing. J. Krutil, Ph.D., Institute of Power Engineering, Brno University of Technology; Technická 2896/2, 616 69 Brno, CZ

Define it from the Navier Stokes equations, where it holds [1], [3], [4]:

$$\varrho\,\frac{\partial v_i}{\partial t} + \varrho\,\frac{\partial}{\partial x_j}(v_i\,v_j) - \frac{\partial \sigma_{ij}}{\partial x_j} = \varrho\,g_i\ . \tag{2}$$

By integration over the volume V and using Gauss-Ostrogradsky theorem we get the force F in the form:

$$F_i = -\varrho\int\limits_V \frac{\partial v_i}{\partial t}\,\mathrm{d}V - \varrho\int\limits_{S\cup\Gamma} v_i\,v_j\,n_j\,\mathrm{d}\Theta + \int\limits_\Gamma \sigma_{ij}\,n_j\,\mathrm{d}\Gamma + \varrho\,g_i\,V\ , \qquad \Theta = S\cup\Gamma\ . \tag{3}$$

From here it is clear that the expression (3) is very difficult to analyze because of the unimaginable estimation of the size of the integral over the volume V.

- The relationship between electric field intensity and transient magnetic field can be difficult to analyze because it is difficult to change the magnetic field inside the region V. It follows from Maxwell's equations, as it applies relations:

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\ . \tag{4}$$

Integration over the volume V can be derived:

$$\varepsilon_{ijk}\int\limits_S E_j\,n_k\,\mathrm{d}S = -\int\limits_V \frac{\partial B_i}{\partial t}\,\mathrm{d}V\ . \tag{5}$$

As in the previous case, the analysis is complicated by the integral over the region V.

The aim of this paper is to modify the mathematical model, so that it was possible to use Gauss-Ostrogradsky methods for the unsteady term integration.

The solution comes out a certain type of operator equations, which is typical for a wide class of the continuum mechanics problems.

2. Mathematical model

Considering multiple contiguous region V bounded by the surface Θ. The boundary orientation is defined by a unit vector outward normal $\mathbf{n}$ to the surface Θ.

The mathematical model is defined by the Cartesian coordinate system, the Euler approach, where each independent variable, generally designated $\mathbf{E}$ depends on the spatial coordinate $\mathbf{x}$ and time t. Thus $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$, $\mathbf{x} = (x_j)$. On the V there is defined the variable field $\mathbf{E}$.

The mentioned area is defined by a mathematical model using the summation convention in the form:

$$\frac{\partial A_i}{\partial t} + \frac{\partial B_{ij}}{\partial x_j} = C_i\ , \quad \mathbf{x}\in V\ , \tag{6}$$

$$\frac{\partial A_i}{\partial x_i} = 0\ , \quad \mathbf{x}\in V\ . \tag{7}$$

In the equations (6), (7), we assume:

$$A_i = A_i(\mathbf{x}, t)\ , \quad B_{ij} = B_{ij}[(\mathbf{x}, t)]\ , \quad C_i = C_i(\mathbf{x}, t)\ , \tag{8}$$

$$\operatorname{div}\mathbf{C} = 0\ . \tag{9}$$

The move to a new model follows; if we apply on the equation (6) the divergence operation and considering (9), we may write:

$$\frac{\partial^2 B_{ij}}{\partial x_i\,\partial x_j} = 0 \qquad \text{or also} \qquad \frac{\partial^2 B_{jk}}{\partial x_j\,\partial x_k}\,x_i = 0\ . \tag{10}$$

After integrating (10) over the region V we obtain:

$$\int\limits_{\Theta} \frac{\partial B_{jk}}{\partial x_k}\,x_i\,n_j\,\mathrm{d}\Theta = \int\limits_{V} \frac{\partial B_{jk}}{\partial x_k}\,\frac{\partial x_i}{\partial x_j}\,\mathrm{d}V\ . \tag{11}$$

Considering $\partial x_i/\partial x_j = \delta_{ij}$, using Gauss-Ostrogradsky theorem we obtain:

$$\int\limits_{\Theta} \left(\frac{\partial B_{jk}}{\partial x_k}\,x_i - B_{ij}\right) n_j\,\mathrm{d}\Theta = 0\ . \tag{12}$$

Expression (12) is very important for the qualitative analysis, it determines the relationship between $\partial B_{jk}/\partial x_k$ and B_{ij} on the boundary Θ, without the influence of unsteady term $\partial A_i/\partial t$: Into (13) we substitute now from (6). The following holds:

$$\int\limits_{\Theta} \left[\left(\frac{\partial A_i}{\partial t} - C_j\right) x_j + B_{ij}\right] n_j\,\mathrm{d}\Theta = 0\ . \tag{13}$$

Expression (13) is crucial for solving of the interactions, because the influence of unsteady term is transformed to the boundary Θ, compared to the original model (6), where for the integration over the region V and the use of Gauss-Ostrogradsky theorem applies:

$$\int\limits_{V} \frac{\partial A_i}{\partial t}\,\mathrm{d}V + \int\limits_{V} \frac{\partial}{\partial x_j} B_{ij}\,\mathrm{d}V = \int\limits_{V} C_i\,\mathrm{d}V\ . \tag{14}$$

Hence it is clear that the analysis and control volume numerical method greatly complicate integrals over the volume V.

To a new mathematical model we move simply using the Gauss-Ostrogradsky theorem on the expression (13). Hence, it is clear that it applies:

$$\int\limits_{V} \frac{\partial}{\partial x_j}\left[\frac{\partial A_j}{\partial t}\,x_i + B_{ij} - C_j\,x_i\right] \mathrm{d}V = 0\ . \tag{15}$$

Considering that the term has to pay for each volume V, can be placed inside the integral term is zero. So shortly:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial A_j}{\partial t}\,x_i + B_{ij} - C_j\,x_i\right) = 0\ . \tag{16}$$

With regard to (7) also

$$\frac{\partial A_j}{\partial x_i} = 0\ . \tag{17}$$

Equations (16), (17) now formulate a new mathematical model for the class of operators of continuum mechanics. The model is not only suitable for the method of control volumes, but especially for the qualitative analysis of the forces interaction within the fields of different physical nature; for example for the determination of the solid/liquid interaction, or a magnetic field.

Evidence of the transition from the model (6) to (18) follows from the following identity; let us put:

$$\frac{\partial A_i}{\partial t} = \frac{\partial}{\partial x_j}(A_j\, x_i) = \frac{\partial A_j}{\partial x_j}\, x_i + A_j\, \frac{\partial x_i}{\partial x_j}\ . \tag{18}$$

Considering the validity of (7) and $\partial x_i/\partial x_j = \delta_{ij}$, the validity of (18) is proved. For the same reason holds $C_i = \partial(C_j\, x_i)/\partial x_j$, since we assume (9).

If is only a function of time, it holds that

$$C_i = C_i(t)\ . \tag{19}$$

Can be used:

$$C_i = \frac{\partial}{\partial x_i}(C_j\, x_j)\ . \tag{20}$$

Based on (13), (14) may have to reformulate the expressions (3), (5).

$$F_i = -\varrho \int_\Theta \left[\frac{\partial v_j}{\partial t}\, x_i + v_i\, v_j - \delta_{ij}\, g_k\, x_k\right] n_j\, \mathrm{d}\Theta + \int_\Gamma \sigma_{ij}\, n_j\, \mathrm{d}\Gamma\ , \tag{21}$$

$$\varepsilon_{ijk} \int_\Theta E_k\, n_j\, \mathrm{d}\Theta = -\int_\Theta \frac{\partial B_j}{\partial t}\, x_i\, n_j\, \mathrm{d}\Theta\ . \tag{22}$$

Or after the treatment

$$\int_\Theta \left[\varepsilon_{ijk}\, E_k - \frac{\partial B_j}{\partial t}\, x_i\right] n_j\, \mathrm{d}\Theta = 0\ . \tag{23}$$

The fundamental significance of these modifications is the possibility to change the unknown quantity on the function area. More important is the possibility to correct formulation of boundary conditions which must be met integral identity.

3. Class problems of continuum mechanics, which can be transferred to the general shape (6), (7), respectively (16), (17)

While solving problems of continuum mechanics we can encounter mathematical models of various physical natures, which have a similar structure. For example:

A. The equation for the vortex velocity $\mathbf{\Omega} = \mathrm{rot}\, \mathbf{v}$ [1], [3]:

$$\frac{\partial \mathbf{\Omega}}{\partial t} + \mathrm{rot}(\mathbf{\Omega} \times \mathbf{v}) = 0\ , \tag{24}$$

$$\mathrm{div}\, \mathbf{\Omega} = 0\ . \tag{25}$$

Comparing with (6), (7) it holds:

$$\mathbf{A} = \mathbf{\Omega}\ , \qquad B_{ij} = \varepsilon_{ijk}\, \varepsilon_{kmn}\, \Omega_m\, v_n\ . \tag{26}$$

B. Maxwell equations : equation for magnetic field strength $\mathbf{H}$ assuming infinite conductivity [2], [3] :

$$\frac{\partial \mathbf{H}}{\partial t} + \mathrm{rot}(\mathbf{H} \times \mathbf{v}) = 0 \ , \tag{27}$$

$$\mathrm{div}\,\mathbf{H} = 0 \ . \tag{28}$$

Comparing with (6), (7) and (24), (25), by analogy it applies :

$$\mathbf{A} = \mathbf{H} \ , \qquad B_{ij} = \varepsilon_{ijk}\,\varepsilon_{kmn}\,H_m\,v_n \ . \tag{29}$$

C. Equations of equilibrium macroscopic particles – Navier-Stokes equations [1], [3] :

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(v_i\,v_j - \frac{\sigma_{ij}}{\varrho}\right) = g_i \ , \tag{30}$$

$$\frac{\partial v_i}{\partial x_i} = 0 \ . \tag{31}$$

Comparing with (6), (7) holds :

$$A_i = v_i \ , \qquad B_{ij} = v_i\,v_j - \frac{\sigma_{ij}}{\varrho} \ , \qquad C_i = g_i \ . \tag{32}$$

D. Homogenous conductor with constant conductivity and permeability

$$\frac{\partial \mathbf{H}}{\partial t} + \mathrm{rot}(\mathbf{H} \times \mathbf{v}) - \alpha\,\Delta\mathbf{H} = 0 \ , \tag{33}$$

$$\mathrm{div}\,\mathbf{H} = 0 \ . \tag{34}$$

$$\mathbf{A} = \mathbf{H} \ , \qquad B_{ij} = \varepsilon_{ijk}\,\varepsilon_{kmn}\,H_m\,v_n - \alpha\,\frac{\partial H_i}{\partial x_j} \ . \tag{35}$$

When we use our model of (16), it's possible to write operators (24), (27), (30), (33) in the shape :

$$\left[\frac{\partial \boldsymbol{\Omega}}{\partial t} + \mathrm{rot}(\boldsymbol{\Omega} \times \mathbf{v})\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial \Omega_j}{\partial t}\,x_i + \varepsilon_{ijk}\,\varepsilon_{kmn}\,\Omega_m\,v_n\right] = 0 \ , \tag{36}$$

$$\left[\frac{\partial \mathbf{H}}{\partial t} + \mathrm{rot}(\mathbf{H} \times \mathbf{v})\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial H_j}{\partial t}\,x_i + \varepsilon_{ijk}\,\varepsilon_{kmn}\,H_m\,v_n\right] = 0 \ , \tag{37}$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(v_i\,v_j - \frac{\sigma_{ij}}{\varrho}\right) - g_i = \frac{\partial}{\partial x_j}\left[\frac{\partial v_j}{\partial t}\,x_i + v_i\,v_j - \frac{\sigma_{ij}}{\varrho} - g_j\,x_i\right] = 0 \ , \tag{38}$$

$$\left[\frac{\partial \mathbf{H}}{\partial t} + \mathrm{rot}(\mathbf{H} \times \mathbf{v}) - \alpha\,\Delta\mathbf{H}\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial H_j}{\partial t}\,x_i + \varepsilon_{ijk}\,\varepsilon_{kmn}\,H_m\,v_n - \alpha\,\frac{\partial H_i}{\partial x_j}\right] = 0 \ . \tag{39}$$

4. Conclusion

From the presented analysis, the usefulness of the Gauss-Ostrogradsky theorem for the qualitative analysis of the mathematical models of continuum mechanics is illustrated. From that theorem, the relation between the changes within the region V on the function values and their gradients acting on the surface S of the region V are defined in a mathematical model. The model can be used for application of the method of finite volumes and qualitative examining of the interactions of the environment with the different physical characteristics. Considering div $\mathbf{A} = 0$ has the original model form:

$$\frac{\partial A_i}{\partial t} + \frac{\partial}{\partial x_j} B_{ij} = C_i \; . \tag{40}$$

It's new, equivalent shape is shown by term:

$$\frac{\partial}{\partial x_j} \left[\left(\frac{\partial A_i}{\partial t} - C_i \right) x_i + B_{ij} \right] = 0 \; . \tag{41}$$

Additionally it holds following identity:

$$\int\limits_V \frac{\partial \mathbf{A}}{\partial t} \, \mathrm{d}V = \int\limits_\Theta \left(\frac{\partial \mathbf{A}}{\partial t} \, \mathbf{n} \right) \mathbf{x} \, \mathrm{d}\Theta \; . \tag{42}$$

Acknowledgement

This work was gratefully acknowledged by grant projects: GAČR P101/13-20031S, CZ.1.07/2.3.00/30.0005

Nomenclature

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}$	– general dependent variable
$\mathbf{x}$	– position vector
t	– time
$\mathbf{n}$	– normal vector
V	– volume
S	– surface
$2\,\boldsymbol{\Omega}$	– angular velocity
$\mathbf{v}$	– velocity vector
δ_{ij}	– Kronecker delta
ε_{ijk}	– Levi-Civit tensor
$\Theta = S \cup \Gamma$	– boundary volume

References

[1] Brdicka M., Samek L., Sopko B.: Mechanika kontinua, In Czech Academia, Prague, 2000, ISBN 80-200-0772-5 (in Czech)

[2] Landau L.D., Lifshitz E.M.: Elektrodynamika splošnych sred, Série 'Teoretická fyzika VIII', Moskva, 2003, p. 656, ISBN 5-9221-0123-4 (T. VIII) (in Russian)

[3] Sedov L.I.: Mechanika splošnych sred, TOM 1, Nauka, Moskava, 1976, p. 536 (in Russian)

[4] Pochyly F., Fialova S., Krausova H.: Variants of Navier-Stokes Equations, In Engineering Mechanics 2012, Book of extended abstracts, Svratka, CZ, Association for Engineering Mechanics 2012, p. 1011–1016, ISBN 978-80-86246-40-6

[5] Pochyly F., Fialova S., Kozubkova M., Bojko M.: Tensors of added damping, stiffness and mass in a thin gap of hydrodynamic machine, IOP Conference Series: Earth and Environmental Science, 2012, 15(7), p. 1–8, ISSN 1755-1315

Received in editor's office: June 25, 2013
Approved for publishing: February 5, 2014