# DYNAMIC RESPONSE OF SOIL-STRUCTURE INTERACTION SYSTEM IN TIME DOMAIN USING ELASTODYNAMIC INFINITE ELEMENTS WITH SCALING MODIFIED BESSEL SHAPE FUNCTIONS

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*This paper is devoted to a new approach to the dynamic response of Soil-Structure System (SSS), the far field of which is meshed by decay or mapped elastodynamic infinite elements, based on scaling modified Bessel shape functions to be calculated. These elements are appropriate for Soil-Structure Interaction problems, solved in time or frequency domain and can be treated as a new form of the recently proposed elastodynamic infinite elements with united shape functions (EIEUSF) infinite elements. Here the time domain form of the equations of motion is demonstrated and used in the numerical example. In the paper only the formulation of 2D horizontal type infinite elements (HIE) is used, but by similar techniques 2D vertical (VIE) and 2D corner (CIE) infinite elements can also be added. Continuity along the artificial boundary (the line between finite and infinite elements) is discussed as well and the application of the proposed elastodynamical infinite elements in the Finite element method is explained in brief. A numerical example shows the computational efficiency and accuracy of the proposed infinite elements, based on scaling Bessel shape functions.*

Keywords: *soil-structure interaction, wave propagation, infinite elements, finite element method, Bessel functions, Duhamel integral*

## 1. Introduction

Infinite elements are widely used in the numerical simulations of engineering problems if unbounded domain exists. Soil-Structure Interaction (SSI) is typical civil engineering problem [1–5, 13–16]. The infinite elements can be integrated in the Finite element method codes [8, 14, 21, 22] adequately dynamic SSI simulations to be obtained. The infinite elements as a computational technology is widely used due to the fact that their concepts and formulations are much closed to those of the finite elements. These elements are very effective for models of structures containing a near field discretized by finite elements and a far field discretized by infinite elements.

The first infinite elements have been proposed in [4] (Bettess) and [20] (Ungless). Classification of the infinite elements is proposed in [7]. During the last three decades much element formulations have been suggested [1, 6, 11, 23, 24, 25]. In the last two decades a lot of dynamic infinite elements were developed, [23, 9, 12].

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## 2. Elastodynamical infinite element with united Bessel shape functions

The idea and concept of the elastodynamic infinite elements with united shape functions (for short EIEUSF class infinite elements) are presented in [10, 11, 12]. Several EIEUSF formulations are discussed and have been demonstrated that the shape functions, related to nodes $k$ and $l$ (the nodes, situated in infinity, fig. 1, are not necessary to be constructed, because corresponding to these shape functions generalized coordinates or weights, see eq. (1), are zeros. The displacements in infinity are vanished, and these shape functions must be omitted. The theory used for the formulation of the EIEUSF class infinite elements has been published in detail in [10], and hence only summarize of the basic idea is demonstrated here. In [11] is mentioned why the EIEUSF class infinite elements are more general and powerful than the standard infinite elements.

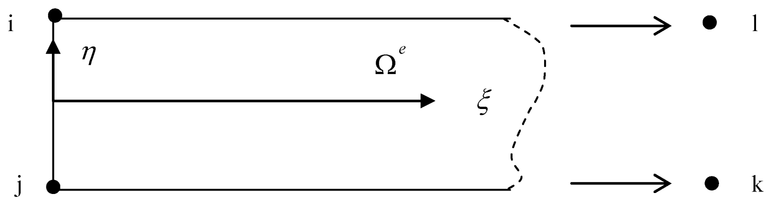


*Fig.1: Local coordinate system of horizontal infinite elements (HIE)*

The displacement field in the elastodynamical infinite element can be described in the standard form of the shape functions based on wave propagation functions as

$$\mathbf{u}(x,z,\omega) = \sum_{i=1}^{n}\sum_{q=1}^{m} N_{iq}(x,z,\omega)\,\mathbf{p}_{iq}(\omega) \qquad \text{or} \qquad \mathbf{u}(x,z,\omega) = N_{\mathrm{p}}(x,z,\omega)\,\mathbf{p}(\omega)\ , \tag{1}$$

where $N_{iq}(x,z,\omega)$ are the standard shape displacement functions, $\mathbf{p}_{iq}(\omega)$ is the generalized coordinate associated with $N_{iq}(x,y,\omega)$, $n$ is the number of nodes for the element and $m$ is the number of wave functions included in the formulation of the infinite element. For horizontal wave propagation basic shape functions for the HIE infinite element, the local coordinate system of which is shown in fig. 1, can be expressed as:

$$N_{iq}(x,z,\omega) = T(x,z,\eta,\xi)\,N_{iq}(\eta,\xi,\omega) = T(x,z,\eta,\xi)\,L_i(\eta)W_q(\xi,\omega)\ , \tag{2}$$

where $W_q(\xi,\omega)$ are horizontal wave functions and $L_i(\eta)$ are Lagrange interpolation polynomial which has unit value at $i$th node while zeros at the other nodes. For HIE infinite element the ranges of the local coordinates are: $\eta \in [-1,1]$ and $\xi \in [0,\infty)$. Here $T(x,z,\eta,\xi)$ assures the geometrical transformations of local to global coordinates.

$$N_i(x,z) = \sum_{q=1}^{m} N_{iq}(x,z,\omega) = T(x,z,\eta,\xi)\,L_i(\eta)\,\operatorname{Re} W(\xi) \tag{3}$$

and

$$N_i(x,z)\,\mathbf{p}_i = \sum_{q=1}^{m} N_{iq}(x,z,\omega)\,\mathbf{p}_{iq}(\omega) = T(x,z,\eta,\xi)\,L_i(\eta)\,\operatorname{Re} W(\xi)\,\mathbf{p}_i\ . \tag{4}$$

Then equation (1) can be expressed as

$$\mathbf{u}(x,z) = N_{\mathrm{p}}(x,z)\,\mathbf{p}\ . \tag{5}$$

For horizontal wave propagation the basic shape functions for the HIE infinite element can be expressed using Bessel functions as follows:

$$N_{iq}(\eta,\xi,\omega) = L_i(\eta)\,\tilde{J}_0^q(\psi\,\xi)\ , \tag{6}$$

where $\tilde{J}_0^q(\psi\,\xi)$ are scaling modified Bessel functions of the first kind. These functions can be written as

$$\tilde{J}_0^q(\psi\,\xi) = J_0^q(\psi\,\xi)\,\exp(-\beta\,\xi)\ , \tag{7}$$

where $J_0^q(\psi\,\xi)$ are standard Bessel functions of first kind. In eq. (6) and (7) $\psi$ and $\beta$ are constants, chosen in such a way that the length of the wave and the attenuation of the wave respectively, are identical with those, if eq. (2) is used. This means that the following two relations are valid:

$$\psi = \frac{\bar{L}_\mathrm{w}}{L_\mathrm{w}} \qquad \text{or} \qquad \hat{\psi} = \frac{\bar{\omega}}{\omega}\ , \tag{8}$$

where $L_\mathrm{w}$ is the wave length if $W_q(\xi,\omega)$ functions are used; $\bar{L}_\mathrm{w}$ – if Bessel functions of first kind $J_0(\xi)$ are used (average distance between two zeros) to approximate the displacements in the infinite element domain, and:

$$\exp(-\beta\,\xi) = \left(\frac{1}{\sqrt{\xi}}\right)^{-1} \exp(-\alpha\,\xi)\ , \tag{9}$$

because the Bessel functions of first kind attenuate proportionally to $1/\sqrt{\xi}$. The zeros of Bessel functions play a dominant role in applications of these functions [17] and demonstrate their oscillatory. Although the roots of Bessel functions are not generally periodic, except asymptotically for large $\xi$, such functions give acceptable results for simulation of wave propagation. And what is more, using Bessel functions one can approximate change of the wave length in the far field region. If the element has four nodes and eight DOF (the simplest two-dimensional plane element [10]) only four shape functions can be used to approximate the displacements, related to one frequency. These functions can be written as:

$$N_{1q}(\eta,\,\xi,\omega) = N_{iq}^u(\eta,\xi,\omega) = L_i(\eta)\,J_0^q(\psi\,\xi)\,\exp(-\beta\,\xi) \qquad \text{or} \tag{10a}$$

$$N_{1q}(\eta,\,\xi,\omega) = N_{iq}^u(\eta,\xi,\omega) = L_i(\eta)\,\tilde{J}_0^q(\psi\,\xi)\ , \tag{10b}$$

$$N_{2q}(\eta,\,\xi,\omega) = N_{iq}^v(\eta,\xi,\omega) = L_i(\eta)\,J_0^q(\psi\,\xi)\,\exp(-\beta\,\xi) \qquad \text{or} \tag{11a}$$

$$N_{2q}(\eta,\,\xi,\omega) = N_{iq}^v(\eta,\xi,\omega) = L_i(\eta)\,\tilde{J}_0^q(\psi\,\xi) \tag{11b}$$

and

$$N_{3q}(\eta,\,\xi,\omega) = N_{jq}^u(\eta,\xi,\omega) = L_j(\eta)\,J_0^q(\psi\,\xi)\,\exp(-\beta\,\xi) \qquad \text{or} \tag{12a}$$

$$N_{3q}(\eta,\,\xi,\omega) = N_{jq}^u(\eta,\xi,\omega) = L_j(\eta)\,\tilde{J}_0^q(\psi\,\xi)\ , \tag{12b}$$

$$N_{4q}(\eta,\,\xi,\omega) = N_{jq}^v(\eta,\xi,\omega) = L_j(\eta)\,J_0^q(\psi\,\xi)\,\exp(-\beta\,\xi) \qquad \text{or} \tag{13a}$$

$$N_{4q}(\eta,\,\xi,\omega) = N_{jq}^v(\eta,\xi,\omega) = L_j(\eta)\,\tilde{J}_0^q(\psi\,\xi) \tag{13b}$$

where in the general case $\xi = \xi_\mathrm{st} + \xi_0$, $\xi_0 \in (0, \bar{L}_\mathrm{w})$.

If rotational DOF are used then the element has four nodes and 1° DOF. Two additional shape functions must be used, written as:

$$N_{5q}(\eta,\,\xi,\omega) = N^{\varphi}_{iq}(\eta,\xi,\omega) = \\ = L_i(\eta)\left[\frac{(J^q_{-1}(\psi\,\xi) - J^q_1(\psi\,\xi))\,\exp(-\beta\,\xi)}{2} - \beta\,J^q_0(\psi\,\xi)\,\exp(-\beta\,\xi)\right] \tag{14}$$

and

$$N_{6q}(\eta,\,\xi,\omega) = N^{\varphi}_{jq}(\eta,\xi,\omega) = \\ = L_j(\eta)\left[\frac{(J^q_{-1}(\psi\,\xi) - J^q_1(\psi\,\xi))\,\exp(-\beta\,\xi)}{2} - \beta\,J^q_0(\psi\,\xi)\,\exp(-\beta\,\xi)\right]\,. \tag{15}$$

Here $J^q_0(\psi\,\xi)$ and $J^q_1(\psi\,\xi)$ are Bessel functions of first kind.

The function $L_i(\eta)$ is linear if no mid-nodes. Finally, if mid-node on the side $i{-}j$ is used, then the Lagrange interpolation polynomials must be quadratic. Scaling modified Bessel functions of first kind, in accordance with eq. (6) ($\tilde{J}^q_0(\psi\,\xi)$ and $\tilde{J}^q_1(\psi\,\xi)$), are illustrated in fig. 2.

The continuity along the artificial boundary (the line between finite and infinite elements, see fig. 3 line $-x_\mathrm{b}$ and line $x_\mathrm{b}$) is assured in the same way as between two plane finite elements [9]. The application of the proposed infinite elements in the Finite element method is discussed below.

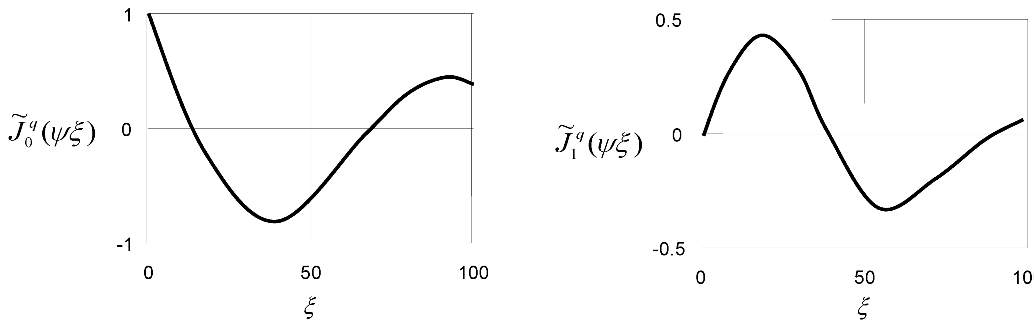


*Fig.2: $\tilde{J}^q_0(\psi\,\xi)$ and $\tilde{J}^q_1(\psi\,\xi)$ scaling modified Bessel functions*

Using the procedure, given in details in [10] and briefly described here, mapped EIEUSF infinite elements, based on scaling modified Bessel functions, can be formulated, based on eq. (16)

$$N_i(x,z) = \sum_{q=1}^{m} N_{iq}(x,z,\omega) = \sum_{q=1}^{m} T(x,z,\eta,\xi)\,N_{iq}(\eta,\xi,\omega) = \sum_{q=1}^{m} T(x,z,\eta,\xi)\,L_i(\eta)\,\tilde{J}^q_0(\psi\,\xi)\,, \tag{16}$$

where $\tilde{J}^q_0(\psi\,\xi) = J^q_0(\psi\,\xi)\,\exp(-\beta\,\xi)$.

## 3. Stiffness and mass matrices

The matrices $\mathbf{K}_{ij}$ and $\mathbf{M}_{ij}$, related to the near field of the Soil-Structure System (SSS) can be written as

$$\mathbf{K}_{ij} = \int\limits_{\Omega_\mathrm{e}} \mathbf{B}^\mathrm{T}_i\,\mathbf{D}\,\mathbf{B}_j\,\mathrm{d}\Omega_\mathrm{e} \tag{17}$$

and

$$\mathbf{M}_{ij} = \left( \int_{\Omega_\mathrm{e}} \varrho \, \mathbf{N}_i^\mathrm{T} \, \mathbf{N}_j \, \mathrm{d}\Omega_\mathrm{e} \right) \mathbf{I} \tag{18}$$

and those related to the far field $\mathbf{K}_\mathrm{b}^\mathrm{g}$ and $\mathbf{K}_\mathrm{b}^\mathrm{g}$, i.e. obtained for the proposed infinite elements, as

$$\mathbf{K}_\mathrm{b}^\mathrm{g} = \int_{\Omega_\mathrm{ie}} \mathbf{B}_i^{(\mathrm{IE})\mathrm{T}} \, \mathbf{D} \, \mathbf{B}_j^{(\mathrm{IE})} \, \mathrm{d}\Omega_\mathrm{ie} \tag{19}$$

and

$$\mathbf{M}_\mathrm{b}^\mathrm{g} = \left( \int_{\Omega_\mathrm{ie}} \varrho \, \mathbf{N}_i^{(\mathrm{IT})\mathrm{T}} \, \mathbf{N}_j^{(\mathrm{IT})} \, \mathrm{d}\Omega_\mathrm{ie} \right) \mathbf{I} \tag{20}$$

where $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{D}$ are shape function matrix, strain-displacement matrix and stress-strain matrix, respectively, and $\mathbf{I}$ is the identity matrix.

The matrices $\mathbf{K}_{ij}$ and $\mathbf{K}_\mathrm{b}^\mathrm{g}$ are calculated using the principle of the virtual work.

If Bessel functions are used, the first derivative of $J_0^q(\psi\,\xi)$ (The Taylor series indicate that by $J_{-1}^q(\psi\,\xi)$ and $J_1^q(\psi\,\xi)$ the derivative of $J_0^q(\psi\,\xi)$ can be expressed) is $\mathrm{d}J_0^q(\psi\,\xi)/\mathrm{d}\xi = = (J_{-1}^q(\psi\,\xi) - J_1^q(\psi\,\xi))/2$.

The general form of the equations of motion in time domain can be written as

$$[M]\,\{\ddot{u}(t)\} + [C]\,\{\dot{u}(t)\} + [K]\,\{u(t)\} = \{f(t)\} \tag{21}$$

where $[M]$, $[C]$ and $[K]$ are mass, damping and stiffness matrices, respectively, and $\{f(t)\}$ is nodal force vector.

The equations of motion of the entire SSS, using the *Substructural* approach with EIEUSF infinite elements, based on scaling modified Bessel functions, transformed into time domain by inverse Fourier transformation, are

$$\begin{bmatrix} \mathbf{M}_\mathrm{ss} & \mathbf{M}_\mathrm{sb} \\ \mathbf{M}_\mathrm{bs} & \mathbf{M}_\mathrm{bb} \end{bmatrix} \left\{ \begin{matrix} \ddot{\mathbf{u}}_\mathrm{s}(t) \\ \ddot{\mathbf{u}}_\mathrm{b}(t) \end{matrix} \right\} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\mathrm{b}^\mathrm{g} \end{bmatrix} \left\{ \begin{matrix} \dot{\mathbf{u}}_\mathrm{s}(t) \\ \dot{\mathbf{u}}_\mathrm{b}(t) \end{matrix} \right\} + \begin{bmatrix} \mathbf{K}_\mathrm{ss} & \mathbf{K}_\mathrm{sb} \\ \mathbf{K}_\mathrm{bs} & \mathbf{K}_\mathrm{bb} + \mathbf{K}_\mathrm{b}^\mathrm{g} \end{bmatrix} \left\{ \begin{matrix} \mathbf{u}_\mathrm{s}(t) \\ \mathbf{u}_\mathrm{b}(t) \end{matrix} \right\} = \\ = \left\{ \begin{matrix} \mathbf{f}_\mathrm{s}(t) \\ \mathbf{f}_\mathrm{b}(t) - \int_0^t \mathbf{S}_\mathrm{b}^\mathrm{g}(t-\tau)\, \exp[-a\,(t-\tau)]\, \mathbf{u}_\mathrm{b}(\tau)\,\mathrm{d}\tau \end{matrix} \right\} \tag{22}$$

if massless far field is assumed. In eq. (21) $\mathbf{u}(t)$ and $\mathbf{f}(t)$ are respectively displacement and force vectors, and $\mathbf{C}_\mathrm{b}^\mathrm{g}$, $\mathbf{K}_\mathrm{b}^\mathrm{g}$ and $\mathbf{S}_\mathrm{b}^\mathrm{g}$ are matrices of mechanical characteristics of the far field soil region. Here

$$\mathbf{f}_\mathrm{b}^\mathrm{g}(t) = \int_0^t \mathbf{S}_\mathrm{b}^\mathrm{g}(t-\tau)\, \exp[-a\,(t-\tau)]\, \mathbf{u}_\mathrm{b}(\tau)\,\mathrm{d}\tau \tag{23}$$

can be assumed as a Duhamel integral or more generally as a convolution integral, for $t \geq \tau$.

Equation (23) is a standard convolution of two functions, given in vector forms, namely $\mathbf{u}_\mathrm{b}(\tau)$ and $\mathbf{f}_\mathrm{b}^\mathrm{g}(t)$. Here the vector components of $\mathbf{u}_\mathrm{b}(\tau)$ can be taken in case of seismic events from seismograms.

If rotational acceleration of the base is possible, than eq. (22) becomes

$$\begin{bmatrix} \mathbf{M}_{\mathrm{ss}} & \mathbf{M}_{\mathrm{sb}} \\ \mathbf{M}_{\mathrm{bs}} & \mathbf{M}_{\mathrm{bb}} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_{\mathrm{s}}(t) \\ \ddot{\mathbf{u}}_{\mathrm{b}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathrm{b}}^{\mathrm{g}} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}_{\mathrm{s}}(t) \\ \dot{\mathbf{u}}_{\mathrm{b}}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{\mathrm{ss}} & \mathbf{K}_{\mathrm{sb}} \\ \mathbf{K}_{\mathrm{bs}} & \mathbf{K}_{\mathrm{bb}} + \mathbf{K}_{\mathrm{b}}^{\mathrm{g}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}}(t) \\ \mathbf{u}_{\mathrm{b}}(t) \end{Bmatrix} = \\ = \begin{Bmatrix} \mathbf{f}_{\mathrm{s}}(t) + \mathbf{f}_{\theta}(t) \\ \mathbf{f}_{\mathrm{b}}(t) - \int_0^t \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(t-\tau)\, \exp[-a\,(t-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau \end{Bmatrix} \tag{24}$$

where $\mathbf{f}_{\theta}(t) = \ddot{\theta}\, h\, \mathbf{m}$.

The matrix $\mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(t-\tau)\, \exp[-a\,(t-\tau)]$ assures the transformation of the nodal unit displacement impulse vector $\hat{\mathbf{u}}_{\mathrm{b}}(\tau)$, applied at moment $\tau$, to a nodal force vector $\hat{\mathbf{f}}_{\mathrm{b}}^{\mathrm{g}}(t)$ at moment $t$ and can be treated as a transformation matrix, the general form of which can be written as $\mathbf{T}(t,\tau)$. This matrix in the present case can be expressed as

$$\mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(t-\tau)\, \exp[-a\,(t-\tau)]\ , \tag{25}$$

where $\mathbf{S}_{\mathrm{b}}^{\mathrm{g}}$ can be treated as a stiffness matrix, the components of which can be calculated from $\mathbf{S}_{\mathrm{b}}^{\mathrm{g}} = \omega^2\, \mathbf{M}_{\mathrm{bb}}^{\mathrm{g}}$.

The vector $\{\mathbf{f}_{\mathrm{b}}(t) - \mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(t)\}$ denotes the vector of interaction forces of the unbounded soil acting at nodes $b$, the nodes situated on the artifitual boundary. These forces are acting as a result of the relative motion between the unbounded soil and the total motion of the near field, see Fig. 3, expressed in vector forms as $\{\mathbf{u}_{\mathrm{b}}(t) - \mathbf{u}_{\mathrm{b}}^{\mathrm{g}}(t)\}$ or $\{\mathbf{u}_{\mathrm{b}}^{\mathrm{t}}(t) - \mathbf{u}_{\mathrm{b}}^{\mathrm{g}}(t)\}$.

For discrete time points the vector $\{\mathbf{f}_{\mathrm{b}}(t) - \mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(t)\}$ is calculated, using eq. (26) in MathCAD.

$$\{\mathbf{f}_{\mathrm{b}}(t) - \mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(t)\} = \left\{ \mathbf{f}_{\mathrm{b}}(t) - \int_0^t \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(t-\tau)\, \exp[-a\,(t-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau \right\}\ . \tag{26}$$

If the force vector, i.e. $\mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(t)$, at moment $\tilde{t}$ is known, i.e.

$$\mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}) = \int_0^{\tilde{t}} \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}-\tau)\, \exp[-a\,(\tilde{t}-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau \tag{27}$$

at moment $\tilde{t} + \Delta t$, the force vector $\mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}+\Delta t)$ can be obtained using

$$\mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}+\Delta t) = \int_0^{\tilde{t}} \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}-\tau)\, \exp[-a\,(\tilde{t}-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau + \\ + \int_{\tilde{t}}^{\tilde{t}+\Delta t} \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}+\Delta t-\tau)\, \exp[-a\,(\tilde{t}+\Delta t-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau \tag{28}$$

or if $\Delta t$ is small time interval using the approximation

$$\mathbf{f}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}+\Delta t) = \int_0^{\tilde{t}} \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}-\tau)\, \exp[-a\,(\tilde{t}-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau + \dot{\mathbf{f}}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t})\, \Delta t = \\ = \int_0^{\tilde{t}} \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\tilde{t}-\tau)\, \exp[-a\,(\tilde{t}-\tau)]\, \mathbf{u}_{\mathrm{b}}(\tau)\, \mathrm{d}\tau + \mathbf{S}_{\mathrm{b}}^{\mathrm{g}}(\Delta t)\, \exp[-a\,(\tilde{t})]\, \mathbf{u}_{\mathrm{b}}(\tilde{t})\, \delta t\ . \tag{29}$$

If eq. (23) is expressed as

$$\mathbf{f}_\mathrm{b}^\mathrm{g}(t) = \int_0^t \mathbf{S}_\mathrm{b}^\mathrm{g}\,\sin[\omega\,(t-\tau)]\,\exp[-\xi\,\omega\,(t-\tau)]\,\mathbf{u}_\mathrm{b}(\tau)\,\mathrm{d}\tau \tag{30}$$

then the trigonometric identity $\sin[\omega\,(t-\tau)] = \sin(\omega\,t)\,\cos(\omega\,\tau) - \cos(\omega\,t)\,\sin(\omega\,\tau)$ can be used and finally

$$\begin{aligned}\mathbf{f}_\mathrm{b}^\mathrm{g}(t) = \cos(\omega\,t)\int_0^t \mathbf{S}_\mathrm{b}^\mathrm{g}\,\sin(\omega\,\tau))\,\exp[-\xi\,\omega\,(t-\tau)]\,\mathbf{u}_\mathrm{b}(\tau)\,\mathrm{d}\tau - {} \\ - \sin(\omega\,t)\int_0^t \mathbf{S}_\mathrm{b}^\mathrm{g}\,\cos(\omega\,\tau))\,\exp[-\xi\,\omega\,(t-\tau)]\,\mathbf{u}_\mathrm{b}(\tau)\,\mathrm{d}\tau\end{aligned} \tag{31}$$

Using the proposed infinite elements, the resulting element stiffness matrices related to the far field are inexpensive to calculate and the global stiffness matrix has relatively small bandwidth. It is reasonable to expect similar results in SSI simulations, based on EIEUSF infinite elements with modified Bessel shape functions to those when EIEUSF infinite elements are used.

The nodal displacement vector at moment $t$ can be calculated using step-by-step method, applied to eq. (23), given in time domain. Such a computational technology is demonstrated in the next Section.

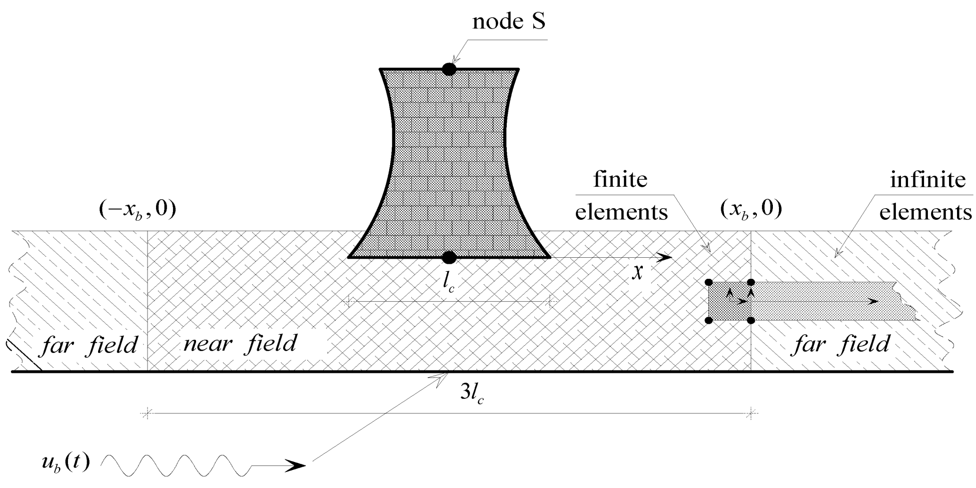


*Fig.3: Computational model*

## 4. Numerical example

Structure with rigid strip foundation resting on a homogeneous half-space is modeled as shown in fig. 3, and the far field is meshed by elastic springs with stiffness , obtained by Tsitovich relation, see [9] (model 1) , by elastic springs with stiffness obtained by Gorbunov-Posadov relation, see [9] (model 2), by massless EIEUSF infinite elements with one wave frequency [9] (model 3) and by massless infinite elements with Bessel shape functions [9] (model 4).

Horizontal harmonic displacements with period $T_\theta = 1\,\mathrm{s}$ and amplitude $u_\mathrm{b}^\mathrm{max} = 0.25\,\mathrm{m}$ are applied on the nodes as shown in fig. 3. The geometry of the model and the material parameters are given in [10]. The basic parameters are: $L = 30\,\mathrm{m}$, $E = 30000\,\mathrm{kN/m^2}$ – structure and $E = 300\,\mathrm{kN/m^2}$ – near and far field.

The results for the first 4 natural periods, corresponding to the models and max displacement of node S, are given in Table 1. The time history of the displacements of node S, see fig. 3, between 9.1 s and 9.5 s are illustrated in fig. 4.

The numerical example shows that, if EIEUSF infinite elements or infinite elements with Bessel shape functions are used, the position of $x_\mathrm{b}$ can be translated starting from $x_\mathrm{b} = 3\,L_\mathrm{c}$ (see $L_\mathrm{c}$ in fig. 3) to $x_\mathrm{b} = L_\mathrm{c}$ without significant influence on the results. However, if elastic springs are used, the results are significantly affected. Such a reduction of the near field demonstrates the effectiveness of the proposed infinite elements.

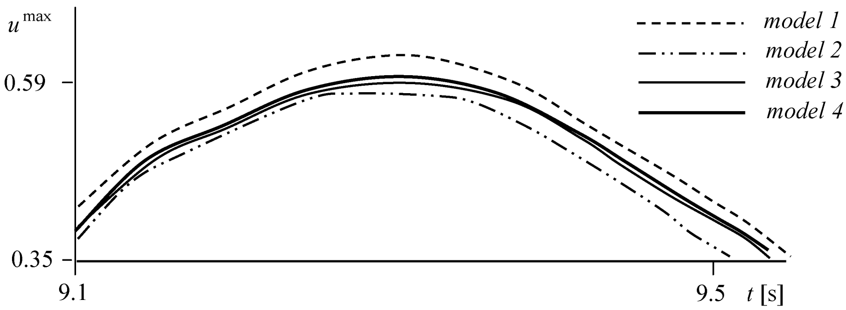


*Fig.4: Time history of the displacements of node S*

| Models | model 1 | model 2 | model 3 | model 4 |
|---|---|---|---|---|
| natural periods of vibration | 1.5628<br>1.5584<br>1.5614<br>1.5615 | 0.7512<br>0.7395<br>0.7455<br>0.7458 | 0.5514<br>0.5377<br>0.5455<br>0.5459 | 0.2278<br>0.1985<br>0.2219<br>0.2239 |
| max displacement [m] | 0.611 | 0.572 | 0.585 | 0.586 |

*Tab.1*

## 5. Conclusion

In this paper a formulation of elastodynamical infinite element, based on scaled Bessel shape functions, appropriate for Soil-Structure Interaction problem, the computational concept and the corresponding equations of motion of the entire SSI system are presented. This element is a new form of the infinite element, given in [9, 10]. The base of the development is new shape functions, obtained by modification of the standard Bessel functions of first kind $J_0(\xi)$ by appropriately chosen scale factor. The stiffness matrices of these infinite elements are calculated by EIEUSF matrix module, developed by the same author.

The numerical example shows the computational efficiency and accuracy of the proposed infinite elements. Such elements can be directly used in the FEM code. The results are in a good agreement with the results, obtained by EIEUSF infinite elements. More over the use of scaling modified Bessel functions in the construction of the shape functions leads to computational efficiency in the stage of the calculation of the stiffness and mass infinite element coefficients.

The formulation of 2D horizontal type infinite elements (HIE) is demonstrated, but by similar techniques 2D vertical (VIE) and 2D corner (CIE) infinite elements can also be formulated. It was demonstrated that the application of the elastodynamical infinite elements is the easier and appropriate way to achieve an adequate simulation (2D elastic media) including basic aspects of Soil-Structure Interaction. Continuity along the artificial boundary (the line between finite and infinite elements) is discussed as well and the application of the proposed elastodynamical infinite elements in the Finite element method is explained in brief.

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**Calculation of of the infinite element, based on scaled Bessel functions**

$$
\begin{aligned}
k_{11} &= \int\limits_{\Omega_{IE}} \frac{\partial}{\partial\xi} N_{1q}(\eta,\xi,\omega)\, D_{11} \frac{\partial}{\partial\xi} N_{1q}(\eta,\xi,\omega)\, \mathrm{d}\Omega_{\mathrm{IE}} = \\
&= D_{11} \int\limits_0^{l_\eta} L_1^2(\eta)\, \mathrm{d}\eta \int\limits_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}\xi} \tilde{J}_0^q(\psi_1\,\xi)\, \frac{\mathrm{d}}{\mathrm{d}\xi} \tilde{J}_0^q(\psi_2\,\xi)\, \mathrm{d}\xi = \\
&= D_{11} \int\limits_0^{l_\eta} L_1^2(\eta)\, \mathrm{d}\eta\, \frac{1}{4} \int\limits_0^{\infty} \Big[ \tilde{J}_{-1}^q(\psi_1\,\xi) + \tilde{J}_1^q(\psi_2\,\xi) \Big] \Big[ \tilde{J}_{-1}^q(\psi_1\,\xi) + \tilde{J}_1^q(\psi_2\,\xi) \Big]\, \mathrm{d}\xi = \\
&= D_{11} \frac{1}{4} \int\limits_0^{l_\eta} L_1^2(\eta)\, \mathrm{d}\eta \int\limits_0^{\infty} \left\{ \Big[ \tilde{J}_{-1}^q(\psi_1\,\xi) \Big]^2 + 2\, \tilde{J}_{-1}^q(\psi_1\,\xi)\, \tilde{J}_1^q(\psi_2\,\xi) + \Big[ \tilde{J}_1^q(\psi_2\,\xi) \Big]^2 \right\} \mathrm{d}\xi = \\
&= \frac{E}{4\,(1-\nu^2)} \int\limits_0^{l_\eta} L_1^2(\eta)\, \mathrm{d}\eta \int\limits_0^{\infty} \Bigg[ \frac{\psi_1^{-1}\, \Gamma\big(\frac{1}{2}(-1)\big)}{\Gamma\big(\frac{1}{2}(-1)+1\big)\, \Gamma(2)}\, {}_2F_1\Big( \big(\tfrac{1}{2}(-1)\big)\,;\, \big(\tfrac{1}{2}(1)\big)\,;\, 2;\, \Big(\tfrac{\psi_1}{\psi_2}\Big)^2 \Big) + \\
&\qquad + 2\, \frac{\psi_1^0\, \psi_2^{-1}\, \Gamma\big(\frac{1}{2}(1)\big)}{\Gamma\big(\frac{1}{2}(-3)+1\big)\, \Gamma(2)}\, {}_2F_1\Big( \big(\tfrac{1}{2}(1)\big)\,;\, \big(\tfrac{1}{2}(3)\big)\,;\, 2;\, \Big(\tfrac{\psi_1}{\psi_2}\Big)^2 \Big) + \\
&\qquad + \frac{\psi_2^{-1}\, \Gamma\big(\frac{1}{2}(3)\big)}{\Gamma\big(\frac{1}{2}(-1)+1\big)\, \Gamma(2)}\, {}_2F_1\Big( \big(\tfrac{1}{2}(3)\big)\,;\, \big(\tfrac{1}{2}(1)\big)\,;\, 2;\, \Big(\tfrac{\psi_1}{\psi_2}\Big)^2 \Big) \Bigg] d\xi\, .
\end{aligned}
$$

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