PARTITIONING OF TWO-DIMENSIONAL NURBS MESHES FOR THE PARALLEL ISOGEOMETRIC ANALYSIS

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Isogeometric analysis is a quickly emerging alternative to the standard, polynomial-based finite element analysis. It is only the question of time, when it will be implemented into major software packages and will be intensively used by engineering community to the analysis of complex realistic problems. Computational demands of such analyses, that may likely exceed the capacity of a single computer, can be alleviated by performing the analyses in a parallel computing environment. However, parallel processing requires usually an appropriate decomposition of the investigated problem to individual processing units. In the case of the isogeometric analysis, the decomposition corresponds to the spatial partitioning of the underlying spatial discretization. While there are several matured graph-based decomposers which can be readily applied to the subdivision of finite element meshes, their use in the context of the isogeometric analysis is not straightforward because of a rather complicated construction of the graph corresponding to the computational isogeometric mesh. In this paper, a new technology for the construction of the dual graph of a two-dimensional NURBS-based (non-uniform rational B-spline) isogeometric mesh is introduced. This makes the partitioning of the isogeometric meshes for parallel processing accessible for the standard graph-based partitioning approaches.

Keywords: *isogeometric analysis, NURBS, parallelization, domain decomposition, dual graph of isogeometric mesh*

1. Introduction

The isogeometric analysis (IGA) builds upon the concept of isoparametric elements and upgrades it to the geometry level. The original intention [1, 2] of the IGA was to span the gap between the computer aided design (CAD) and the finite element method (FEM) by eliminating the need to have a separate representation for the original CAD model and yet another one for the actual computational geometry. This is achieved by using the same basis functions for the description of the geometry and for the approximation of the solution space on that geometry. As a consequence, the isogeometric mesh (discretization for computational purposes) of the CAD geometry encapsulates the exact (up to the CAD representation) geometry no matter how coarse the mesh actually is. This attractive feature together with the range of applicability and other advantages make the IGA an interesting alternative to the widely used FEM. The properties and computational performance of the IGA has been studied elsewhere [1, 3, 4, 5, 6] showing a clear evidence that the IGA may outperform the classical FEM in various aspects (accuracy, robustness, system condition number, etc.).

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Solution of complex engineering problems using the IGA or FEM may lead to computationally very extensive as well as intensive task, demands of which can be alleviated by performing it in a parallel computing environment. Typical parallel application decreases the demands on memory and other resources by spreading the analysis over several mutually interconnected computers and speeds up the response of the application by distributing the computation to individual processors. Note that parallelization of the analysis can be often the only way how to make solution of large scale problems accessible in reasonable time or even at all.

There are many similar features between the IGA and the FEM [1, 2] which allow to adopt the same parallelization concepts and paradigms. From the computational costs point of view, however, one can identify important differences having impact on the overall performance. On one side, the computational costs of the IGA are shifted from the global computation (solution of the overall system of equations which is typically smaller compared to that of the FEM) toward the local computation (numerical quadrature of characteristic vectors and matrices on the element (knot span) level, which is typically more demanding compared to the FEM). On the other side, the bandwidth of the resulting system of equations in the IGA is wider, which is a direct consequence of the fact that the support of the B-spline based basis functions of quadratic or higher degree is larger (than just a single patch of elements sharing a node). This implies that from the computational costs point of view, the IGA is more appropriate for the parallel processing than the FEM, because the local computation can be easily parallelized while the global computation not. On the other hand, the performance of the parallel analysis is generally influenced also by communication costs which are dependent (not only) on the amount of data transferred between computational units involved in the parallel processing, which is in turn dependent on the number of nodes (or elements) shared between individual processors. Due to the structure of the B-spline basis functions, the number of the shared nodes (elements) is larger (relatively to the overall number of nodes (elements) of the problem) in the IGA and quickly grows with the increasing degree of basis functions. This makes the properly balanced partitioning of the computational isogeometric mesh a key ingredient for the efficient parallel processing of the IGA. However, compared to the FEM, the decomposition of the isogeometric mesh is rather non-trivial. The problem consists in the fact that the construction of the graph representing properly the isogeometric mesh is not straightforward, which is the consequence of the relation between the computational (non-zero) knot spans and control points in the isogeometric mesh. This paper introduces a novel concept how to construct the dual graph of the NURBS-based isogeometric mesh that can be decomposed by standard graph-based partitioning methods [7, 8, 9, 10, 11].

The paper is organized as follows. The concept of the IGA using the NURBS is firstly briefly recalled in Section 2. The detailed description of the construction of the dual graph of the NURBS-based isogeometric mesh is presented in Section 3. In Section 4, a few examples of dual graphs of various two-dimensional NURBS-based meshes and their decompositions are provided. And finally, the paper ends with concluding remarks in Section 5.

2. NURBS-based isogeometric analysis

In the IGA, the approximation of the solution over the domain is based on the functions employed for the description of the underlying geometry of the domain itself. Therefore understanding of the NURBS based representation of the geometry (used in CAD) gives a good insight into the isogeometric concept.

A NURBS patch is defined by a set of control points (topologically forming a regular grid of the dimension corresponding to the spatial dimension of the underlying parametric space), their weights, degree of the B-spline basis functions in each direction of the parametric space, and a so-called knot vector represented by a non-decreasing sequence of parametric coordinates for each direction defining the support for individual B-spline basis functions in that particular direction. Note that the number of control points, degree of basis functions, and size of the knot vector in the particular parametric direction are not independent and must be mutually consistent. The data at the control points (for example the coordinates when the geometry is concerned, or the primary unknowns when the solution space is handled) are spread over the NURBS patch using the shape functions which are defined as weighted normalized tensor product of uni-variate B-spline basis functions in each of the parametric directions. For example, for a two-dimensional NURBS patch of degree p in u-direction and degree q in v-direction, the basis function associated with control point in the i-th row and the j-th column of the grid of $M \times N$ (rows $\times$ columns) control points is given by

$$R^{p,q}_{i,j}(u,v) = \frac{N^p_j(u)\, N^q_i(v)\, w_{i,j}}{\sum_{m=1}^{M} \sum_{n=1}^{N} N^p_n(u)\, N^q_m(v)\, w_{m,n}} , \tag{1}$$

where $w_{i,j}$ stands for the control point weight and $N^r_k(t)$ denotes the uni-variate B-spline basis functions of degree r. Starting with the piecewise constant basis functions of zero degree defined by

$$N^0_i(t) = \begin{cases} 1 & \text{if} \quad t_i \le t < t_{i+1} \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{2}$$

the basis functions for degree $p > 0$ are defined recursively as

$$N^p_i(t) = \frac{t - t_i}{t_{i+p} - t_i}\, N^{p-1}_i(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}}\, N^{p-1}_{i+1}(t) \ , \tag{3}$$

in which t_i (for $i = 1, 2, \ldots, N+p+1$) stands for entries of the knot vector and N denotes the number of control points (in the given direction). This is demonstrated in Figure 1a where the cubic basis function N^3_i spanning 4 (i.e. degree + 1) consecutive knot spans is obtained as a linear combination of consecutive quadratic basis functions N^2_i and N^2_{i+1} spanning the first three and the last three from those 4 knot spans, respectively. Figure 1b then displays the hierarchical sequence for piecewise constant, linear, and quadratic basis functions built over an infinite uniform knot vector. For details concerning the definition of the B-spline basis functions and their properties the reader is referred to [12].

An example of a quadratic NURBS curve (i.e. one-dimensional NURBS patch) defined by six control points and their weights and parametrized over the open knot vector[1] $\{0, 0, 0, 1, 3, 3, 4, 4, 4\}$ is depicted in Figure 2a. The parametric equation of that particular curve is given by

$$\mathbf{r}(t) = \sum_{i=1}^{6} R_i(t)\, \mathbf{P}_i \ , \tag{4}$$

[1] Knot vector is called open if its first and last entry is repeated (degree + 1) times, which implies that the curve is passing through the first and last control point (see [12] for details).

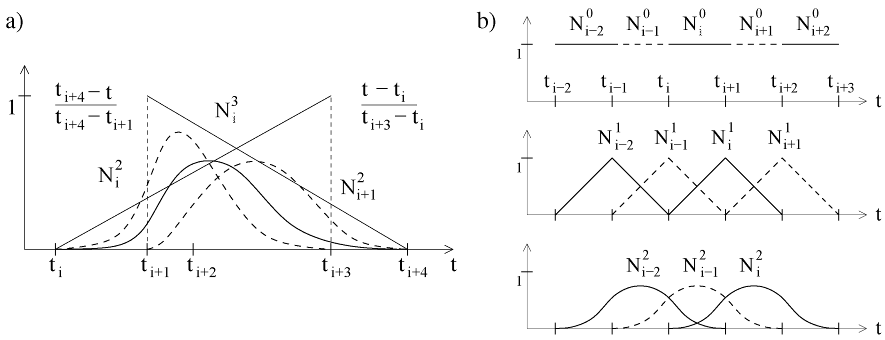

Fig.1: B-spline basis functions: (a) construction of cubic basis function as linear combination of quadratic basis functions, (b) hierarchical sequence of piecewise constant, linear, and quadratic basis functions on uniform knot spans

where $\mathbf{r}$ is the positional vector of a point on the curve corresponding to parameter $t \in \langle 0, 4\rangle$ and $\mathbf{P}_i$ represents the individual control points. The NURBS basis functions $R_i^2(t)$ as well as the B-spline basis functions $N_i^2(t)$ used to construct $R_i^2(t)$ are shown in Figure 2b over the entire span of the knot vector. The curve interpolates those control points for which the corresponding basis function attains value one, the rest of the control polygon is only approximated. Note, however, how the curve is attracted toward legs of control polygon at control point P_3 possessing higher weight. The curve is C^1 continuous everywhere except for the point corresponding to parameter 3 (control point P_4) at which the continuity has been weakened by repeating that particular value in the knot vector twice[2]. Note the C^0 continuity of the B-spline basis function N_4 in Figure 2b at parameter 3. Since basis functions, continuity of which has been reduced to C^0 (or C^{-1}), attain value of one at the corresponding knot, the curve is passing also through control point P_4.

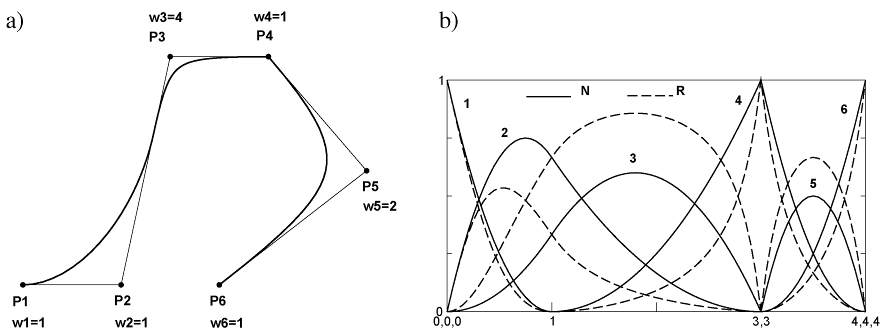

Fig.2: Quadratic NURBS curve: (a) control polygon with weights, (b) B-spline basis functions N_i and NURBS basis functions R_i corresponding to individual control points plotted over the entire span of the knot vector $\{0, 0, 0, 1, 3, 3, 4, 4, 4\}$

The computational isogeometric mesh within the single NURBS patch is formed by partitioning the parametric space into the non-zero knot spans in each direction (in the example above, there are three such non-zero knot spans, see Figure 2). Since the shape functions within the single non-zero knot span are C^∞, the computation of characteristic

[2] Generally, multiplicity $k \leq p$ of a particular inner knot decreases the continuity of the basis functions of degree p at that knot to C^{p-k}.

components of the discretized governing differential equation (e.g. stiffness matrix, load vector, etc.) on each non-zero knot span is performed in the standard FE-like fashion, typically using the Gaussian numerical quadrature[3].

3. Construction of dual graph of isogeometric mesh

Similarly to the FEM, the parallelization strategy of the IGA is based on the (spatial) domain decomposition concept. Within the context of the FEM, there are generally two dual partitioning approaches. With respect to the character of a cut, dividing the computational mesh into partitions, one can distinguish between node-cut and element-cut strategies [14]. While in the node-cut strategy (see Figure 3a), the elements are uniquely assigned to individual partitions by leading the cut between elements across (so called shared) nodes, in the element-cut approach (see Figure 3b), the nodes are uniquely assigned to individual partitions by running the cut across (also so called shared) elements.

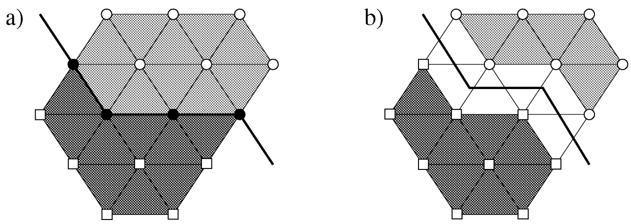

Fig.3: Partitioning of a finite element mesh: (a) node-cut approach; local elements (corresponding to different partitions) in gray, empty circles and squares correspond to local nodes on different partitions, filled circles correspond to shared nodes, (b) element-cut approach; local elements (corresponding to different partitions) in gray, shared elements are empty, empty circles and squares correspond to local nodes on different partitions; cut is indicated in thick solid black line

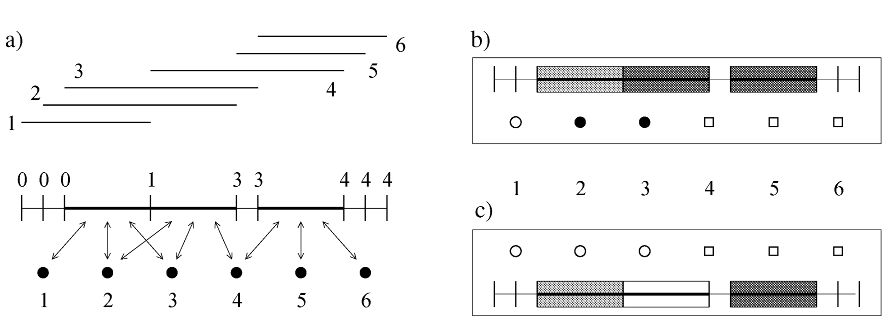

Fig.4: Partitioning of the isogeometric mesh of a quadratic curve: (a) relation between non-zero knot spans and support of basis functions corresponding to individual control points, (b) node-cut approach; local knot spans (corresponding to different partitions) in gray, empty circles and squares correspond to local control points on different partitions, filled circles correspond to shared control points, (c) element-cut approach; local knot spans (corresponding to different partitions) in gray, shared knot spans are empty, empty circles and squares correspond to local control points on different partitions

[3] Note, however, that Gaussian numerical quadrature is generally not optimal and there exist more efficient numerical integration schemes for the IGA [13].

In the context of the IGA, the identification of shared nodes (control points) and shared elements (non-zero knot spans) is not so obvious due to the rather complex relation between the control points and non-zero knot spans. The difference between the node-cut and element-cut applied to the isogeometric mesh of the quadratic NURBS curve from Figure 2 is shown in Figure 4. The relation between the non-zero knot spans and the support of basis functions associated with individual control points (compare with Figure 2b) is schematically depicted in Figure 4a. In the node-cut approach (see Figure 4b), the non-zero knot spans are uniquely assigned to individual partitions. The control points on the patch are then classified according to the support of corresponding basis functions. While control points basis functions of which are non-zero only on local (on-partition) spans are classified as local, control points with non-zero basis functions on spans assigned to different partitions are classified as shared. In the element-cut strategy (see Figure 4c), the control points are uniquely assigned to individual partitions. The non-zero knot spans are then classified with respect to the locality of the support of basis functions of individual control points. Those non-zero knot spans which form support for basis functions corresponding to control points assigned to different partitions are classified as shared knot spans, the other ones as local. Note that in order to keep the patch definition complete, on each partition owning at least a part of the patch, not only the local and shared nodes, but also the remaining (called remote) control points have to be stored. However, there are no unknowns at the remote control points.

In the context of the FEM, the node-cut approach is generally considered more efficient due to the smaller amount of inter-processor data transfer and because the duplicated processing of shared elements is avoided. For the IGA, there is not yet enough evidence available to make such a conclusion. However, since the computation on the non-zero knot spans of the isogeometric mesh is dominant, one may expect, that the node-cut strategy will be again the better one. This is the reason why only the node-cut strategy is considered thereafter in this paper (without explicitly mentioning this fact).

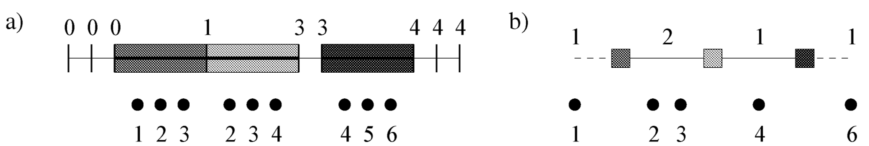

Fig.5: Dual graph of the isogeometric mesh of a quadratic curve: (a) knot vector of the curve with non-zero spans distinguished by different levels of gray; basis functions with support on a particular non-zero knot span are listed below the knot spans, (b) dual graph of the mesh; graph vertices correspond to individual non-zero knot spans, cut of a particular graph edge results in shared control points (enumerated below the graph edges) number of which is displayed above the graph edges; ordinary edges are indicated in solid line, interface edges in dashed line

The partitioning approaches [7, 8, 9, 10, 11] for finite element meshes are typically based on the subdivision of a graph representing the mesh. In the case of the node-cut strategy, the mesh is represented by a dual graph in which the vertices of the graph correspond to mesh elements and the graph edges represent the connectivity of elements. There may be assigned weights to vertices as well as edges of the graph. While the weight of graph vertices corresponds to the relative computational costs (on elements), the weight of graph edges is associated with the communication costs. Since the communication cost is related to the amount of transferred data which is dependent on the number of shared nodes, the edge weight should reflect the number of shared nodes due to cutting that edge. For finite element meshes composed of elements of the same type, the edge weight can be usually neglected. In such a case, the partitioning algorithm attempts to minimize the nominal interface which is proportional to the actual number of shared nodes.

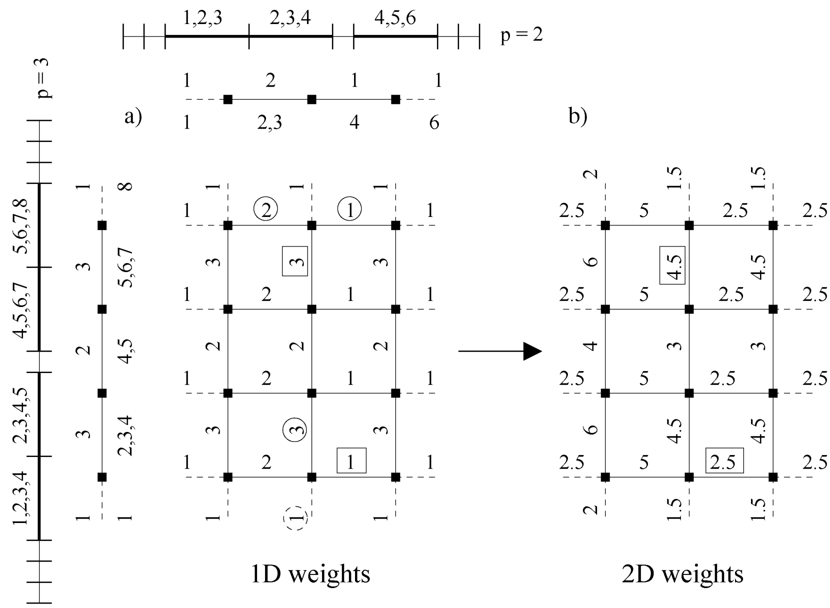

Fig.6: Construction of the dual graph of the isogeometric mesh of a quadratic × cubic surface: (a) dual graph with one-dimensional weights setuped for quadratic ($p = 2$) curve in the horizontal direction (displayed on the top) and cubic ($p = 3$) curve in the vertical direction; non-vanishing basis functions on individual non-zero knot spans are shown above knot spans, shared control points below the graph edges, and weights above the graph edges, (b) dual graph with two-dimensional weights recalculated from one-dimensional weights; examples of recalculation formulas: $3\times(2/2+1/2)=4.5$, $1\times(1+3/2)=2.5$

In the case of the isogeometric mesh, the vertices of its dual graph are formed by non-zero knot spans and the edges correspond to their connectivity. From the above example of the node-cut partitioning of the one-dimensional isogeometric mesh, it is apparent that the number of shared control points (for a particular cut) is given by the number of basis functions which are non-zero on two consecutive non-zero knot spans[4] as it is demonstrated in Figure 5. Note that in order to enable connection of the curve to adjacent curves (on either side), interface edges (indicated in dashed lines) with weight 1 have been introduced in the graph, because (as a consequence of using open knot vector) only the single boundary control point needs to be shared with the adjacent curve. If two graphs are connected the corresponding interface edges are merged into single ordinary edge using the smaller from the weights on the merged interface edges. The weight w_{t_i} of an ordinary graph edge

[4] In this context, two non-zero knot spans are considered consecutive even if there is any number of zero knot spans between them.

corresponding to distinct knot t_i is given by

$$w_{t_i} = p + 1 - k_{t_i} \ , \tag{5}$$

where k_{t_i} denotes the multiplicity of that knot and p stands for the degree. In order to generalize the above relationship also for interface edges, it is modified to

$$w_{t_i} = \max(1, p + 1 - k_{t_i}) \ . \tag{6}$$

Since the graph edge weight is generally dependent on the degree of basis functions and on the knot vector (more precisely on the multiplicity of inner knots) and because both these quantities may be different for each direction of a multi-dimensional patch, it is obvious that in the context of an isogeometric mesh, the weights of its dual graph edges are essential for balanced partitioning minimizing the number of shared control points.

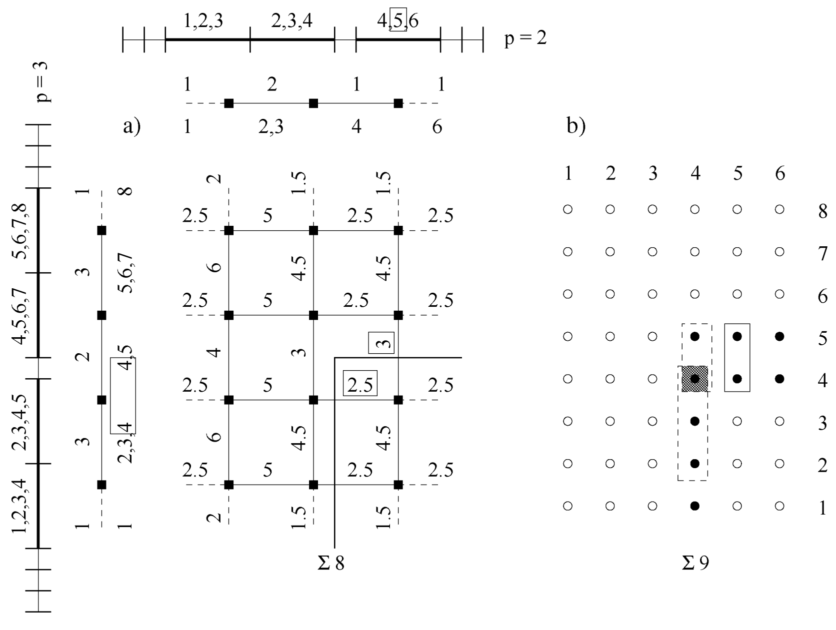

Fig.7: Partitioning of the isogeometric mesh of a quadratic × cubic surface : (a) dual graph of the isogeometric mesh with two-dimensional weights; cut is indicated in thick solid line, (b) grid of control points; black control points are shared with respect to the cut; control point inside of the overlap (indicated in gray) of dashed boxes is duplicated, control points inside of the solid box are missing in the sum of weights of cut graph edges

When constructing the dual graph of the isogeometric mesh of a multi-dimensional NURBS patch, the determination of the weights of the graph edges is slightly complicated by the fact that also the number of shared control points in the direction(s) not perpendicular to the cut must be accounted for. This is illustrated in Figure 6 which clarifies the setup of graph edge weights for a quadratic × cubic NURBS surface. The two-dimensional weight of a particular graph edge is obtained by multiplying the corresponding one-dimensional weight by the linear combination of weights of perpendicular graph edges connected to the

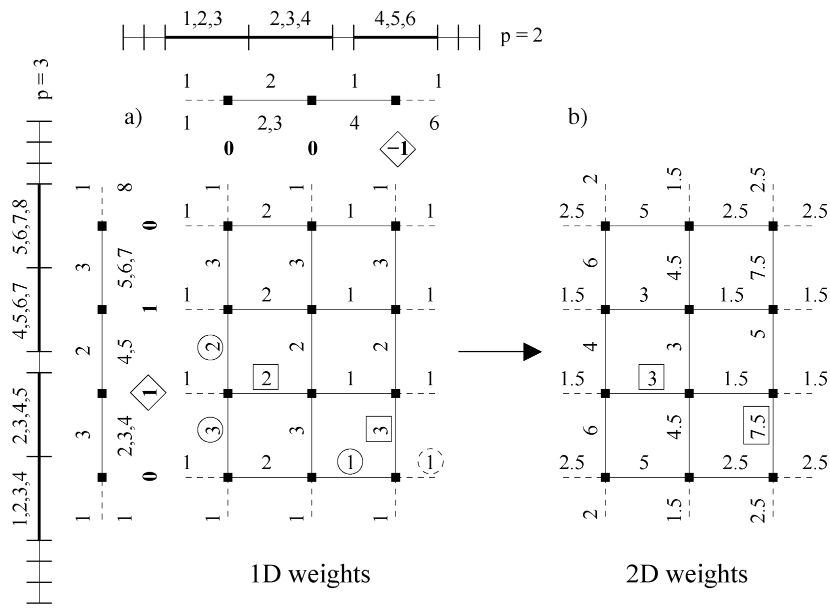

Fig.8: Corrected construction of the dual graph of the isogeometric mesh of a quadratic × cubic surface : (a) dual graph with one-dimensional weights setuped for quadratic ($p = 2$) curve in the horizontal direction and cubic ($p = 3$) curve in the vertical direction; data from Figure 6 are supplemented by the number of duplicated shared control points shown in bold beneath the vertices of one-dimensional graphs, (b) dual graph with corrected two-dimensional weights recalculated from one-dimensional weights; examples of recalculation formulas : 2×(3/2+2/2-1)=3, 3×(1+1/2+1))=7.5

processed graph edge at either end (it does not matter at which one). In the linear combination, the weight of the interface edge is taken by full value, the weight of the ordinary edge by half of the value (the weight contributes to perpendicular graph edges at both end vertices). However, this simple recalculation of one-dimensional weights to two-dimensional weights does not reflect the fact that some of the shared control points corresponding to consecutive non-zero knot spans are the same and/or that some control points do not manifest themselves in the weights of the graph. This is demonstrated in Figure 7 where the isogeometric mesh of the same quadratic × cubic surface is subdivided by a cut resulting in 9 shared control points. However, the sum of weights of the cut graph edges amounts only to 8. The problem is that control points 4 and 5 in the cubic direction (see Figure 7a) are duplicated in the enumeration of shared control points. This implies that a vertical cut through horizontal edges corresponding to the two middle non-zero knot spans in the cubic direction counts the control points 4 and 5 twice. On the contrary, control point 5 in the quadratic direction is not shared at all. This is quite correct for one-dimensional case, in which the quadratic basis function 5 corresponding to the control point 5 is non-zero only on a single non-zero knot span. In the two-dimensional case, however, any horizontal cut crossing the edge corresponding to that knot span results in the underestimation of the number of shared control points. Therefore the number of duplicated and missing shared control points for each non-zero knot span have to be included in the evaluation of the weights of the graph. Note that the number of missing shared control points can be treated as the negative number of duplicated shared control points (and vice versa). The number of duplicated shared control points, d, is given for each non-zero knot span by

$$d = p - 1 - (\min(k_{t_{\text{left}}} - 1, p - 1) + \min(k_{t_{\text{right}}} - 1, p - 1)) \ , \tag{7}$$

where $k_{t_{\text{left}}}$ and $k_{t_{\text{right}}}$ denote the multiplicity of knots bounding the knot span from left and right, respectively. The corrected two-dimensional weight of a particular graph edge is obtained by multiplying the corresponding one-dimensional weight by the linear combination of weights of perpendicular graph edges connected to the processed edge at either end (it does not matter at which one) reduced by the number of duplicated shared control points. The final process of the calculation of corrected two-dimensional weights of edges of the dual graph of the isogeometric mesh of a NURBS surface is presented in Figure 8[5]. Note, however, that the dual graph with the corrected edge weights ensures proper assessment of the number of shared control points only under the assumption, that the cut does not have parallel sections too close to each other. The reason for this fact is that the number of duplicated shared control points in a particular direction is reflected only by the cuts parallel with that direction and does account for the duplication of shared control points for perpendicular cuts close enough to each other, as it is revealed in Figure 9. Although a posteriori correction of the total number of shared control points due to the above effect can be easily derived for a particular cut, there is no way how to modify the weights of individual edges of the dual graph to reflect that effect in general.

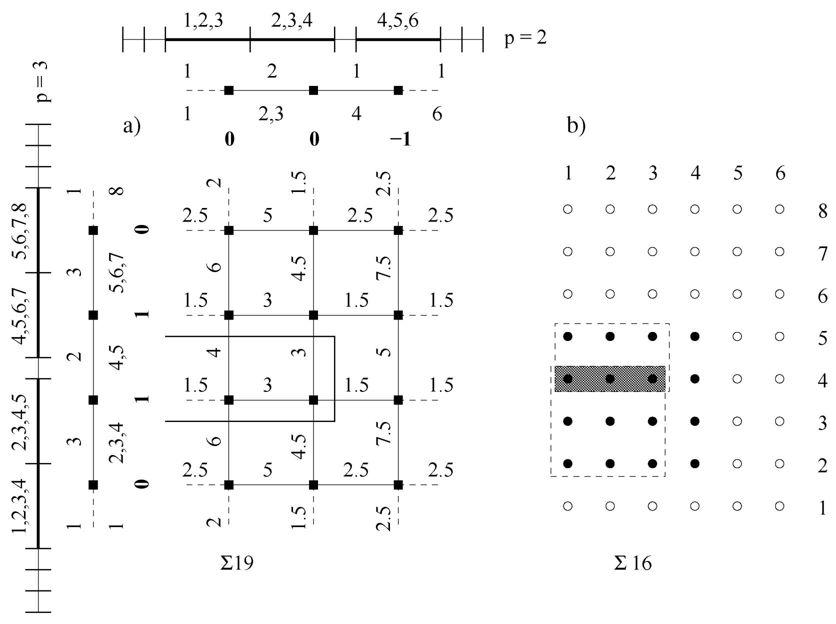

Fig.9: Partitioning of the isogeometric mesh of a quadratic × cubic surface : (a) dual graph of the isogeometric mesh with corrected two-dimensional weights; cut is indicated in thick solid line, (b) grid of control points; black control points are shared with respect to the cut; control points inside of the overlap (indicated in gray) of dashed boxes are duplicated in the sum of weights of cut graph edges

[5] Check, that the sum of weights of graph edges with respect to the cut from Figure 7 yields correct number of the shared nodes, is left on the reader.

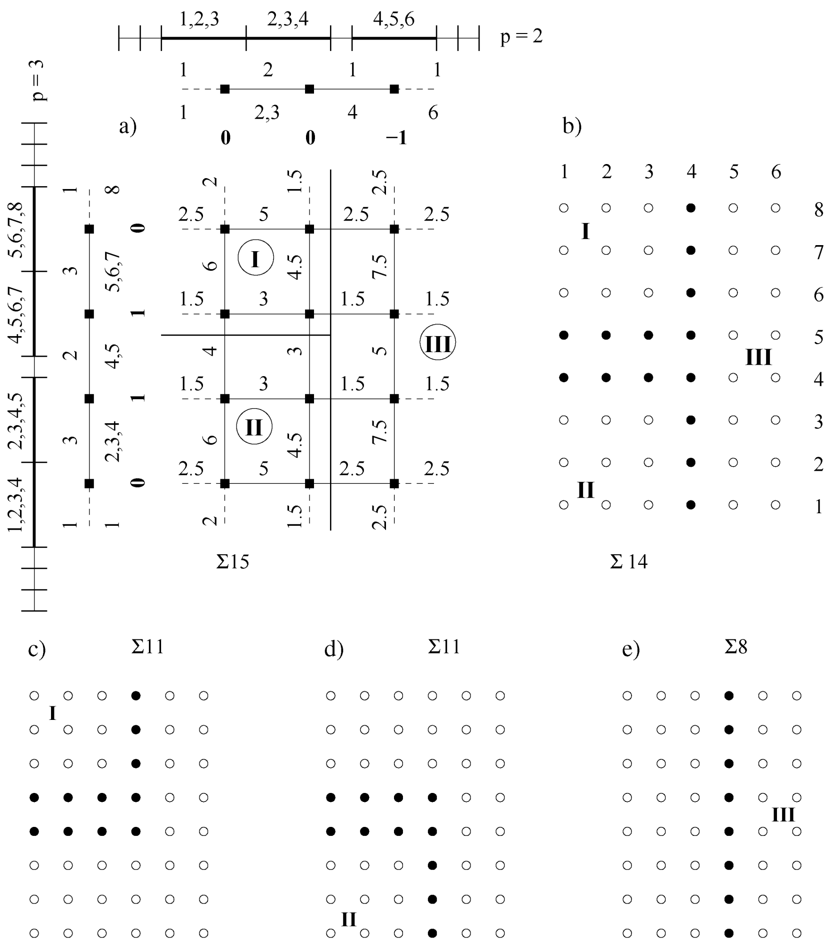

Fig.10: Partitioning of the isogeometric mesh of a quadratic × cubic surface to three subdomains : (a) dual graph of the isogeometric mesh; cut is indicated in thick solid line, individual partitions by roman numbers, (b) shared control points corresponding to the partitioning to three subdomains, (c) shared control points corresponding to the interface along partition I, (d) shared control points corresponding to the interface along partition II, (e) shared control points corresponding to the interface along partition III

In the above, only the decomposition to two partitions has been considered. In the case of partitioning the isogeometric mesh to more than two subdomains, the number of shared control points assessed as the sum of weights of cut graph edges is larger than the real number of shared control points if at least one control points is shared by more than two partitions. This is demonstrated in Figure 10 in which the partitioning of the isogeometric surface mesh to three partitions (see Figure 10a) results in 14 shared control points (see Figure 10b), which is less than 15 estimated from the dual graph. This can be explained by the fact that there is not precise correspondence between a particular graph edge and a set of control points. Therefore, while the cut can be uniquely split to distinct parts corresponding to the interface between pairs of adjacent partitions, the same cannot be done with related shared control points. Since the overall cut is obtained as additive merge of its distinct parts, the sum of weights of the cut graph edges is generally not the same as the total number of shared control points. This is illustrated in Figures 10c,d,e in which the shared control points corresponding to cuts separating the individual partitions from the rest of the domain are highlighted[6]. For each of these cuts the real number of shared control points is in agreement with the number of shared control points derived from the dual graph. By additive merging all these three cuts, the original partitioning to three subdomains is obtained with the overall sum of weights of cut graph edges equal to 30, which is double[7] of the sum of weights in the original decomposition. Since the sets of shared control points in individual decompositions overlap only partially each other (not each shared control point is used twice), their additive merging cannot be performed and simple union, leading to (correct) 14 shared control points, must be adopted.

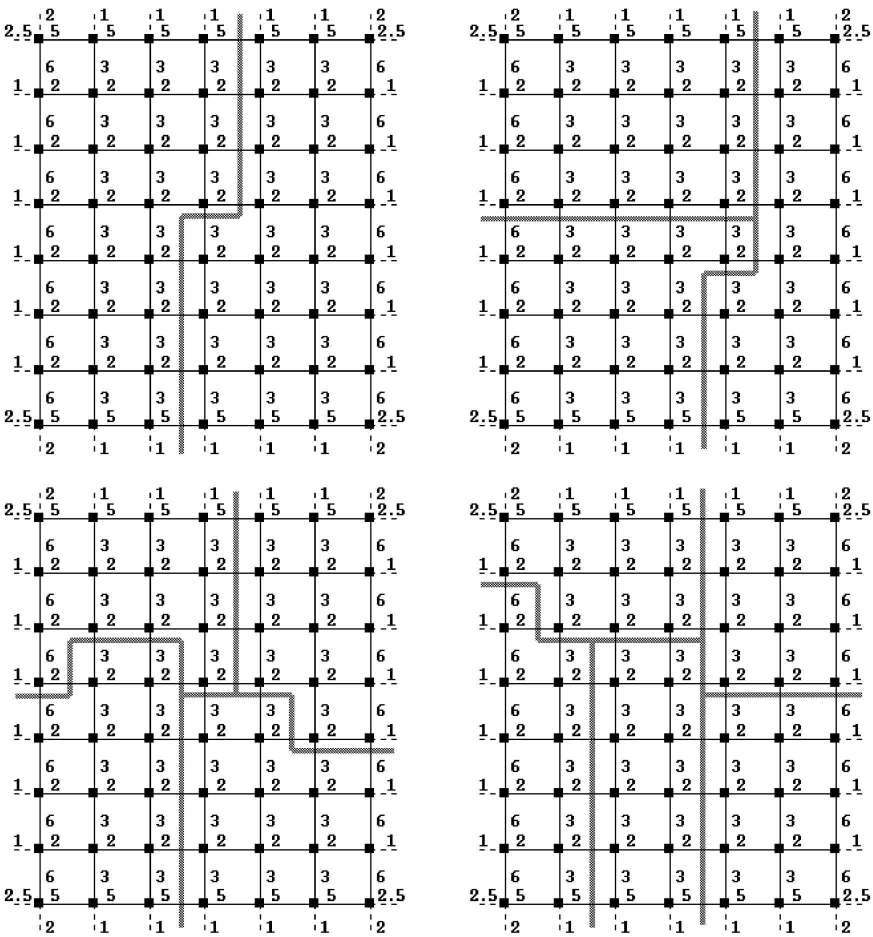

Fig.11: Partitioning of the dual graph of surface A to 2 (top left), 3 (top right), 4 (bottom left), and 5 (bottom right) subdomains; the subdomain interface is indicated in gray

[6] Note that parts of the overall cut corresponding to interface between pairs of adjacent partitions cannot be represented on the set of control points.

[7] Note that each cut graph edge is used twice.

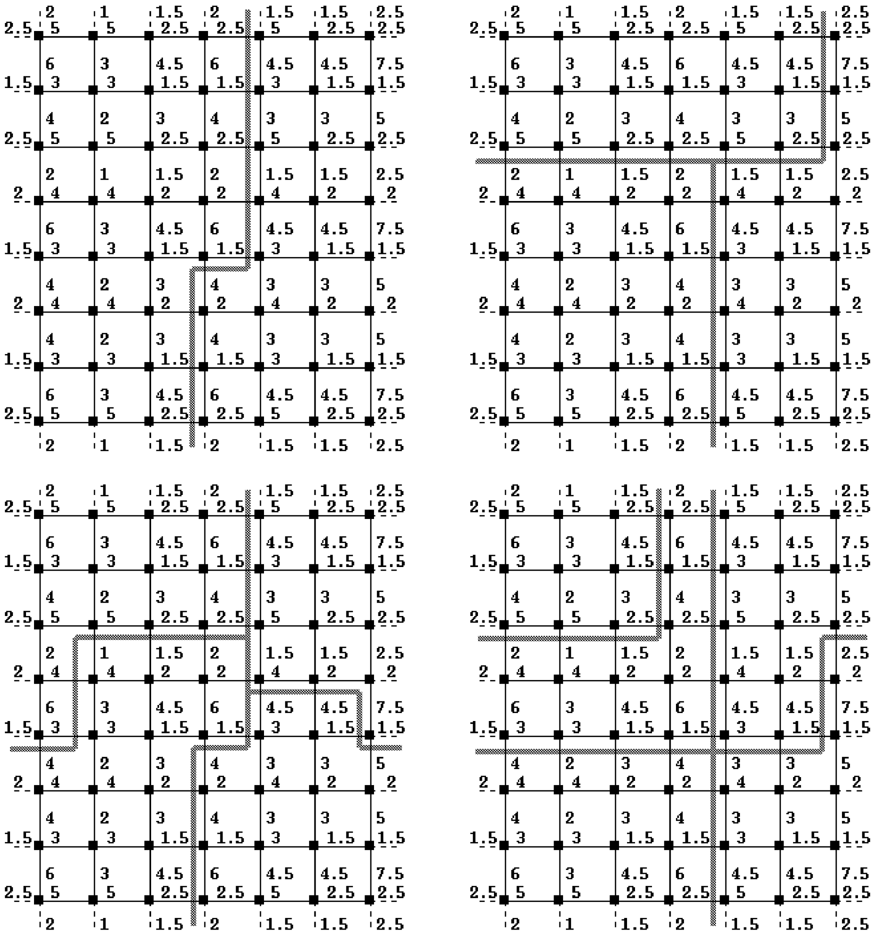

Fig.12: Partitioning of the dual graph of surface B to 2 (top left), 3 (top right), 4 (bottom left), and 5 (bottom right) subdomains; the subdomain interface is indicated in gray

surface id	deg	spans	knot multiplicity	pnts	deg	spans	knot multiplicity	pnts
	in u-direction (horizontal)				in v-direction (vertical)			
A	2	7	3 1 1 1 1 1 1 3	9	3	8	4 1 1 1 1 1 1 1 4	11
B	2	7	3 1 1 2 2 1 2 3	12	3	8	4 1 2 2 1 3 2 1 4	16
C	5	7	6 1 1 1 1 1 1 6	12	2	8	3 1 1 1 1 1 1 1 3	10
D	5	7	6 1 4 1 2 2 1 6	17	2	8	3 1 2 2 1 1 1 2 3	13

Tab.1: Relevant parameters of investigated isogeometric surfaces : degree, number of non-zero spans, multiplicity of distinct knots, and number of control points for each parametric direction

4. Examples

The application of the described construction of the weighted dual graph is presented on the partitioning of isogeometric meshes of a few surfaces, relevant parameters of which are summarized in Table 1. Surfaces A and C are defined by knot vectors with multiplicity of inner knots equal to 1. However, while the degrees of surface A in individual parametric

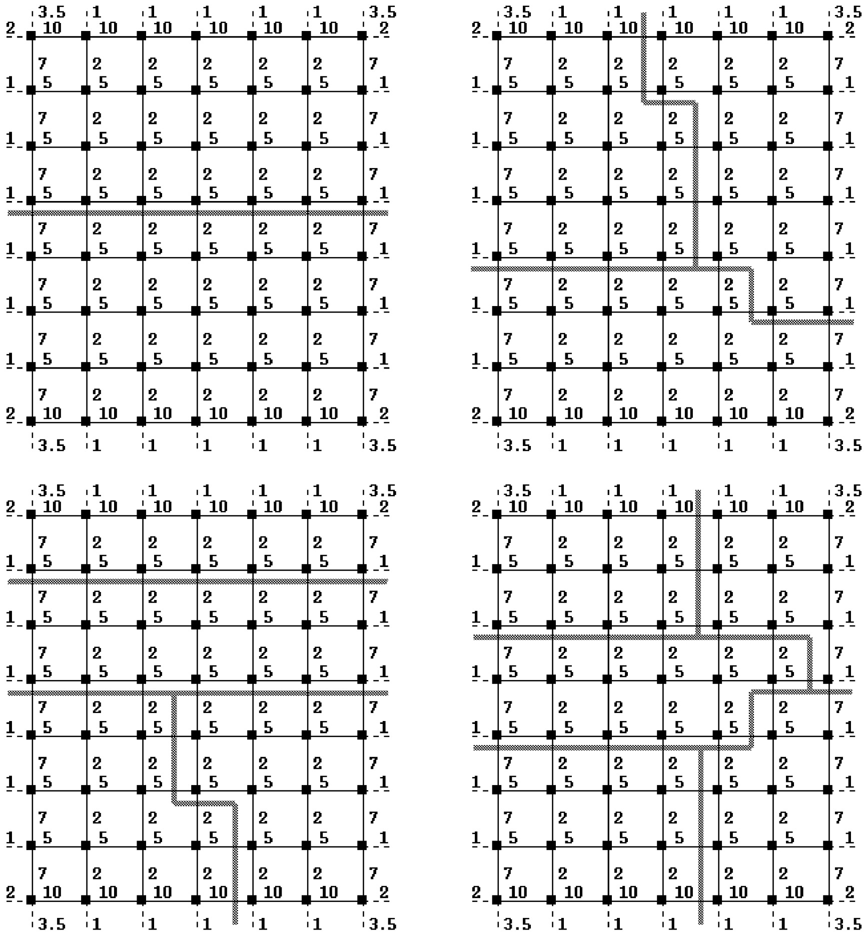

Fig.13: Partitioning of the dual graph of surface C to 2 (top left), 3 (top right), 4 (bottom left), and 5 (bottom right) subdomains; the subdomain interface is indicated in gray

directions are similar, the degrees of surface C are quite different, which makes one direction more preferable for leading the cut. Surfaces B and D are then variants of surfaces A and C with variable multiplicity of inner knots. The number of non-zero spans was chosen for all surfaces the same, which emphasizes the role of the degree and knot multiplicities in the decomposition. The decomposition of individual surfaces to 2, 3, 4, and 5 subdomains is depicted in Figures 11–14. The partitioning was performed using the Metis 4.1 graph partitioner [11] applied to the dual graph of the isogeometric mesh of the surface with unit weight assigned to the vertices of the graph. The parameters of the resulting decompositions are summarized in Table 2. The discrepancy between the real and estimated number of shared control points is caused by both factors discussed at the end of Section 3, this means by parallel cuts too close to each other and by cuts resulting in control points shared by three and more subdomains. The results reveal, hoverer, that the decompositions are well balanced and the differences between the real and estimated number of shared control points are not of significant importance (with the only exception of partitioning of surface C to 5 subdomains). The agreement between the real and estimated number of shared control

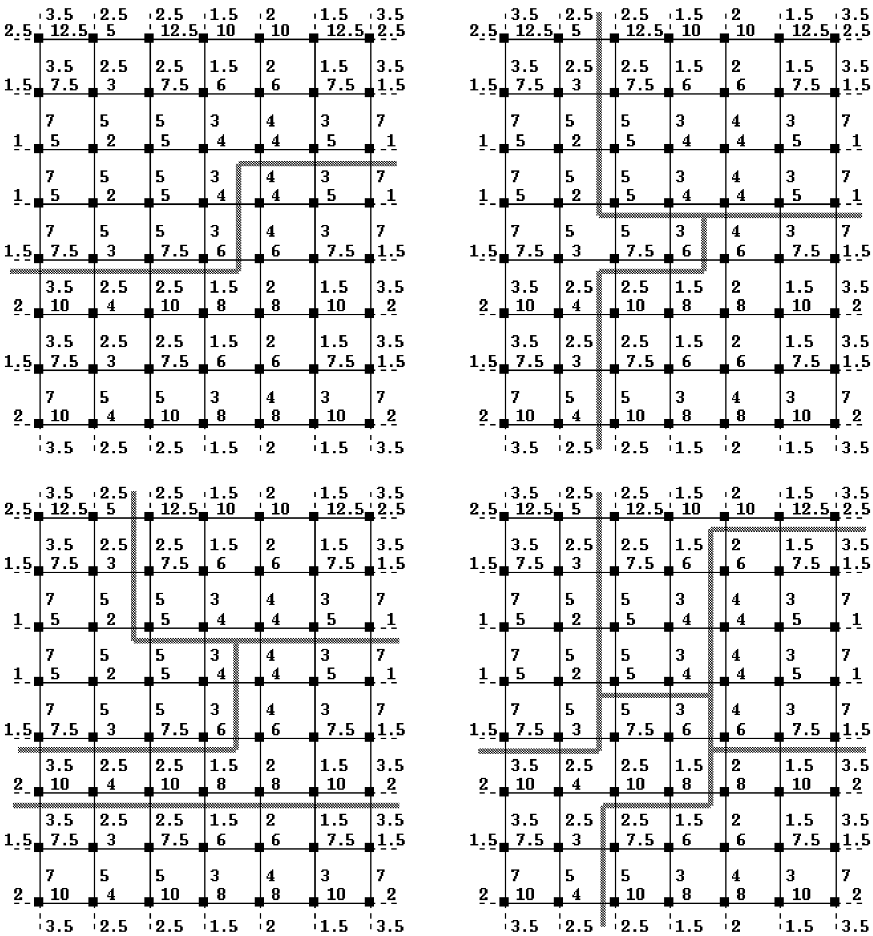

Fig.14: Partitioning of the dual graph of surface D to 2 (top left), 3 (top right), 4 (bottom left), and 5 (bottom right) subdomains; the subdomain interface is indicated in gray

surface	A	B	C	D
number of parts	real / estimated number of shared control points			
	number of non-zero knot spans on individual partitions			
2	25 / 25	20 / 20	24 / 24	34 / 34
	28 28	27 29	28 28	27 29
3	38 / 43	25 / 25.5	56 / 61	51 / 55
	17 19 20	18 18 20	18 19 19	18 18 20
4	48 / 55	49 / 51	70 / 75	65 / 69
	14 14 14 14	13 14 14 15	14 14 14 14	13 14 14 15
5	55 / 64	50 / 52	75 / 91	64 / 82
	10 11 11 12 12	9 11 11 12 13	10 11 11 12 12	10 10 11 12 13

Tab.2: Parameters of the produced decompositions : ratio of the real and estimated number of shared control points and the balancing of individual partitions in terms of non-zero knot spans

points is generally better for the surfaces with knot vector of higher multiplicity of inner knots. This was expected because with increasing multiplicity of inner knot, the number of shared control points is decreasing and the number of overlapping shared control points in the set 'corresponding' to distinct parts of the cut is decreasing as well. It is also interesting to note considerable sensitivity of Metis to the choice of parametric directions. By swapping u- with v-direction (without changing any other parameter) of the surface, which actually corresponds just to the 'transposition' of its parametric space without influencing its shape, completely different decomposition can be obtained even for just two subdomains. This is probably caused by the Metis' heuristics used to coarsen the graph before the actual decomposition is performed and then again by the heuristics applied during fine balancing of the decomposition within the inverse refinement process. This implies that the presented decompositions are not necessarily the best that can be achieved.

5. Conclusions

The present paper introduces a novel methodology how to represent the two-dimensional NURBS-based isogeometric mesh by a dual graph. This enables to use standard graph-based partitioning approaches to obtain its balanced decomposition with minimized communication costs, which can be adopted for parallel isogeometric analysis. The performance of the presented algorithm has been demonstrated on several single patch based examples proving the vitality of the proposed approach. The developed strategy is applicable not only to the decomposition of a single patch but can be generalized also to a set of patches with compatible parameterization (at least in terms of the number of non-zero knot spans of the compatible size) along the interface between them. To account for possibly various computational costs on individual patches (for example due to various degree), the dual graph can be extended by the weights of its vertices corresponding to the computational costs on individual non-zero knot spans. Although one can admit that for a large enough set of patches, a reasonably computationally balanced decomposition can be obtained on the level of patches (along interfaces between patches) without actually subdividing the individual patches, the authors believe that similar methodology could be applied to the partitioning of T-spline based surfaces [15] that have the potential to describe complex domains by just a single surface, partitioning of which is the only way how to accomplish appropriate spatial decomposition.

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References

[1] Hughes T.J.R., Cottrell J.A., Bazilevs Y.: Isogeometric Analysis: CAD, Finite Elements, NURBS, Exact Geometry and Mesh Refinement, Computer Methods in Applied Mechanics and Engineering, 194 (2005), pp. 4135–4195

[2] Cottrell J.A., Hughes T.J.R., Bazilevs Y.: Isogeometric Analysis: Toward Integration of CAD and FEA, John Wiley & Sons 2009

[3] Cottrell J.A., Reali A., Bazilevs Y., Hughes T.J.R.: Isogeometric Analysis of Structural Vibrations, Computer Methods in Applied Mechanics and Engineering, 195 (2006), pp. 5257–5296

[4] Cottrell J.A., Hughes T.J.R., Reali A.: Studies of Refinement and Continuity in Isogeometric Structural Analysis, Computer Methods in Applied Mechanics and Engineering, 196 (2007), pp. 4160–4183
[5] Auricchio F., da Veiga L.B., Buffa A., Lovadina C., Reali A., Sangalli G.: A Fully 'Locking-free' Isogeometric Approach for Plane Linear Elasticity Problems: A Stream Function Formulation, Computer Methods in Applied Mechanics and Engineering, 197 (2007), pp. 160–172
[6] Lipton S., Evans J.A., Bazilevs Y., Elguedj T., Hughes T.J.R.: Robustness of Isogeometric Structural Discretizations under Severe Mesh Distortion, Computer Methods in Applied Mechanics and Engineering, 199 (2010), pp. 357–373
[7] Walshaw C., Cross M.: Multilevel Mesh Partitioning for Heterogeneous Communication Networks, Future Generation Computer Systems, 17 (2001), pp. 601–623
[8] Schloegel K., Karypis G., Kumar V.: Graph Partitioning for High Performance Scientific Simulations, CRPC Parallel Computing Handbook, Morgan Kaufmann 2001
[9] Schloegel K., Karypis G., Kumar V.: Parallel Static and Dynamic Multi-Constraint Graph Partitioning, Concurrency and Computation: Practice and Experience, 14 (2002), pp. 219–240
[10] Boman E., Devine K., Fisk L.A., Heaphy R., Hendrickson B., Leung V., Vaughan C., Catalyurek U., Bozdag D., Mitchell W.: Zoltan home page, http://www.cs.sandia.gov/Zoltan, 1999
[11] Karypis G.: METIS home page, http://www.cs.umn.edu/~metis, 1998
[12] Piegl L., Tiller W.L.: The NURBS Book, Springer-Verlag, 1997
[13] Hughes T.J.R., Reali A., Sangalli G.: Efficient Quadrature for NURBS-based Isogeometric Analysis, Computer Methods in Applied Mechanics and Engineering, 199 (2010), pp. 301–313
[14] Krysl P., Bittnar Z.: Parallel Explicit Finite Element Solid Dynamics with Domain Decomposition and Message Passing: Dual Partitioning Scalability, Computers and Structures, 79 (2001), pp. 345–360
[15] Sederberg T.W., Zheng J., Bakenov A., Nasri A.: T-splines and T-NURCCs, ACM Transactions on Graphics (SIGGRAPH 2003), 22 (2003), pp. 477–484

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