STOCHASTIC PROGRAMMING MODELS FOR ENGINEERING DESIGN PROBLEMS

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The purpose of the paper is to introduce various stochastic programs and related deterministic reformulations that are suitable for engineering design problems. Firstly, several application areas of engineering design are introduced and cited. Then, motivation ideas and basic concepts are presented. Later, various types of reformulations are introduced for decision problems involving uncertainty. In addition, short notes on comparison of optimal solutions are included.

Keywords: underlying program, stochastic program, deterministic reformulation, comparisons

1. Introduction

Many optimization problems belong to the area of engineering design. Quite a lot of them can be modeled by mathematical programs. Recently studied problems frequently include uncertain parameters, a complicated structure, and the need to find optimal or suboptimal values of design variables. They can be found, e.g., in traditional mechanical or civil engineering design problems, see [43] and [15] or in the design of transportation and generally logistic networks [23] or in the problems of plant parameter design, see [24], [25], [7] and [29]. In the case of design problems, the uncertain parameters can be modeled by random elements (material characteristics, loads, etc.) must also be considered, see, e.g., [42] for reliability computations; [21], [22] for hedging against failures of steel production system; [14] and [41] for various concrete design problems and considered uncertain loads. Many optimum design problems in civil and mechanical engineering lead to optimization models constrained by differential equations, see [10] and [3]. In general, these problems can be modeled by using stochastic optimal control formulations. However, the discussion about these problems identified their stage-related decision structure and the need for robust modeling and solution approaches i.e. there is a need among engineers to allow significant changes of the model without necessity to choose or develop another algorithm (e.g., in the case of additional constraints and modified terms in the model). Hence, because of these requests and our experience, we mostly suggest and utilize scenario-based stochastic programs.

While mathematical programs constrained by PDE are frequently studied and applied in deterministic situations, see [10], stochastic programming was firstly used for the case of shape optimization by [6]. The development of suitable computational schemes for above mentioned application areas had began a few yeas ago at the Brno University of Technology (BUT) by Žampachová, Popela et al., see [28] for modeling, [46] and [47] for various cases. Firstly, stochastic programs constrained by hyperbolic and parabolic PDEs were built and

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solved, see Žampachová [48] for details. Then, the case of a random load on the beam has been studied in [49]. These problems lead to large-scale optimization models involving random parameters, see [4] and [38]. Therefore, the efficient decomposition techniques (PHA) have been developed by Wets and Rockafellar, see [36] and implemented at the BUT, see [49] and [16]. The applicability of suitable decomposition algorithms [32] and heuristic algorithms regarding scenario generation [37] and global optimum search for nonlinear cases is further studied, see [20] and [19].

There are often discussions among engineers that for the given optimization problem there must be the best model found. However, modelers may follow various optimization paradigms (e.g., model uncertainty by using either stochastic or fuzzy concepts), choose different levels of abstraction (e.g., time continuous or discretized), define various approximations of the ideal model (e.g., nonlinear convex or its piece-wise approximation), have distinct optimization goals (cf. multicriteria optimization), and even their understanding of the request 'find an optimal solution' can be diverse (e.g., search for the global optimum contrary to the enough good feasible solution). Therefore, the same problem is modeled in different ways. Fortunately, the recent development of hardware power and software flexibility allows modelers to utilize multimodeling ideas instead of guessing, which of the model alternatives to choose as the best one. Hence, we list several general models for the aforementioned problems inspired by [34].

2. Basic Concepts

Traditional deterministic decision problems in engineering are often modeled by so called mathematical programs, see [1] and [2].

Definition 1 (Mathematical program) We define mathematical program (MP) as:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ f(\mathbf{x}) \mid \mathbf{x} \in C \} ,$$
 (1)

where $C \subseteq \mathbb{R}^n$ is called a feasible set, $n \in \mathbb{N}$, $f : C \to \mathbb{R}$ is called an objective function, and $\mathbf{x} \in C$ describes a decision (vector) variable.

Many decision problems are modelled as MPs. MPs (1) often involve important constant (deterministic) parameters. We may emphasize this fact by writing parameters explicitly in the MP formulation. Applications of optimization also show that it is important to use more than one MP. So, the approach of using more than one model is called *multimodeling*. Multiplicity of models can also be formally introduced by parametric MPs as follows.

Definition 2 (Parametric MP) We define parametric MP (PMP) as:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ f(\mathbf{x}, \mathbf{a}) \mid \mathbf{x} \in C(\mathbf{a}) \} ,$$
 (2)

where $\mathbf{a} \in \mathbb{R}^{K}$ is a constant parameter, $K \in \mathbb{N}$.

So, the complex application problems can be modeled in various ways and also different optimal solutions are obtained. The natural question appears: 'Which of these optimal solutions is the most suitable one?' In the rare case, we may be lucky and the optimal solution will be the same for various programs. However, we have to consider a frequent case when the optimal solution changes with the model change (described by the change of parameter \mathbf{a}).

We have already mentioned that many real-world complex problems include various levels of uncertainties, and they can be modeled in different ways [13]. One of them is stochastic optimization. We concentrate on the cases where the program parameters are influenced by random variables. The probabilistic interpretation of the uncertain parameters in the mathematical program is known as *stochastic programming*. The mathematical program involving random parameters is called an *underlying program* [13]. From the parametric mathematical program, we obtain an underlying program by replacing some constant parameters by random variables. We ask what is the meaning of the underlying program. It is understood as a syntactically correct description, for which semantics is given later.

Definition 3 (Underlying program) We define an *underlying (stochastic) program* (UP) as:

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$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ f(\mathbf{x}, \boldsymbol{\xi}) \mid \mathbf{x} \in C(\boldsymbol{\xi}) \} ,$$

where $\boldsymbol{\xi}: \Omega \to \mathbb{R}^{K}$ is a random vector, for (Ω, \mathcal{F}, P) given probability space. As $\boldsymbol{\xi}$ is an \mathcal{F} -measurable mapping, it induces a probability distribution on \mathbb{R}^{K} . We denote a probability space as $(\mathbb{R}^{K}, \mathcal{B}, \mathcal{P})$ or $(\Xi, \mathcal{B}, \mathcal{P})$, where Ξ is a support of \mathcal{P} (the smallest set by \subseteq such that $\mathcal{P}(\Xi) = 1$). \mathcal{B} is a σ -field of Borel's sets on \mathbb{R}^{K} . Derived probabilities are computed by the rule $\forall B \in \mathcal{B}: \mathcal{P}(B) := P(\{\omega \mid \boldsymbol{\xi}(\omega) \in B\})$. Because of \mathcal{F} -measurability $\{\omega \mid \boldsymbol{\xi}(\omega) \in B\} \in \mathcal{F}$, and hence, \mathcal{P} domain is specified fully and consistently. To emphasize the use of random vector $\boldsymbol{\xi}$, we often write $P(\boldsymbol{\xi} \in B)$ instead of $\mathcal{P}(B)$. $P(\boldsymbol{\xi} \in B)$ is a short version of $P(\{\omega \mid \boldsymbol{\xi}(\omega) \in B\})$. $\forall \omega^{s} \in \Omega: \boldsymbol{\xi}(\omega^{s}) \in \mathbb{R}^{K}$ is a realization (observation) of $\boldsymbol{\xi}$. In short, we write $\boldsymbol{\xi}^{s}$.

Although the UP description is syntactically correct, it is unclear from the semantical point of view. Therefore, we have to introduce its *deterministic reformulation* that correctly interprets the presence of random parameters. One of the important questions is whether the deterministic reformulation can be expressed in an *explicit form* of a traditional mathematical program. Such an explicit form is often called an *algebraic equivalent* and we suggest to focus on it in case of using discretization. All these programs that involve random parameters in syntactically correct ways are called *stochastic programs*, see [28].

3. Objective Function Deterministic Reformulations

The main further question that should be answered is when the decision will be made – before the random parameters are observed or after the observations are known. According to Madansky [13], when the decision is made after the observing the randomness, this case is called a *wait-and-see* (WS) approach. This approach is valuable when we know the realization of $\boldsymbol{\xi}$ before making our decision, and it assumes the perfect information about the future. In this case, we may modify our decision by observation, and hence, the decision \mathbf{x} is a function $\mathbf{x}(\boldsymbol{\xi})$ of the random vector $\boldsymbol{\xi}$. Also, the outcome $f(\mathbf{x}(\boldsymbol{\xi}); \boldsymbol{\xi})$ is a random variable.

Definition 4 (WS deterministic reformulation) Let the UP (see Definition 3) be given. We define its wait-and-see (WS) deterministic reformulation:

?
$$\in \underset{\mathbf{x}(\boldsymbol{\xi})}{\operatorname{argmin}} \{ f(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi}) \mid \mathbf{x}(\boldsymbol{\xi}) \in C(\boldsymbol{\xi}) \} ,$$
 (4)

where $\mathbf{x}(\boldsymbol{\xi})$ denotes a (measurable) mapping $\mathbf{x} : \mathbb{R}^K \to \mathbb{R}^n$. We emphasize the relation of the solution to WS reformulation (4) by using superscripts WS i.e. $\mathbf{x}_{\min}^{WS}(\boldsymbol{\xi})$ and $z_{\min}^{WS}(\boldsymbol{\xi})$.

We denote the set of all optimal solutions by $X_{\min}(\boldsymbol{\xi})$ (or $X_{\min}^{WS}(\boldsymbol{\xi})$) i.e.

$$X_{\min}(\boldsymbol{\xi}) = \underset{\mathbf{x}(\boldsymbol{\xi})}{\operatorname{argmin}} \{ f(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi}) \mid \mathbf{x}(\boldsymbol{\xi}) \in C(\boldsymbol{\xi}) \} .$$
(5)

Therefore, WS deterministic reformulation (4) could be interpreted as a special parametric mathematical program (cf. $(2)^1$) with 'probability-based weights'. To emphasize the fact that not only one \mathbf{x}_{\min} is computed, we may also write:

$$\forall \boldsymbol{\xi} \in \Xi : \ ? \in \underset{\mathbf{x}(\boldsymbol{\xi})}{\operatorname{argmin}} \{ f(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi}) \mid \mathbf{x}(\boldsymbol{\xi}) \in C(\boldsymbol{\xi}) \} \ . \tag{6}$$

In the WS case we apply sensitivity analysis ideas, we study how the optimal solution changes when it is allowed to follow the changes of model parameters or a model structure or even a qualitatively different model.

Decision makers must often make decisions before the observations of random parameters are known. In this case, they are using a so called *here-and-now* (HN) approach. The decision \mathbf{x} must be the same for any future realization of $\boldsymbol{\xi}$. Stochastic programming deals primarily with here-and-now decisions, because the typical decision situation is described by the lack of observations. The first general idea for the HN case is to utilize knowledge we already have i.e. to apply one of known WS solutions for certain realization $\boldsymbol{\xi}^s$ i.e. to choose $\mathbf{x}_{\min}(\boldsymbol{\xi}^s)$. We will further denote $\mathbf{x}_{\min}(\boldsymbol{\xi}^s)$ shortly as \mathbf{x}_{\min}^s . Those realizations $\boldsymbol{\xi}^s$ are often called *scenarios* in application problems. So, we solve the problem for one selected scenario.

Definition 5 (IS deterministic reformulation) Let the UP (see Definition 3) be given. We define its here-and-now individual scenario (IS) deterministic reformulation (IS program):

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ f(\mathbf{x}, \boldsymbol{\xi}^s) \mid \mathbf{x} \in C(\boldsymbol{\xi}^s) \} ,$$
 (7)

where $\boldsymbol{\xi}^s \in \Xi$ is a specified individual scenario. We denote the minimal objective function value as z_{\min}^{IS} and minimum as $\mathbf{x}_{\min}^{\text{IS}}$.

To learn how good is the obtained solution, it is reasonable to evaluate $f(\mathbf{x}, \boldsymbol{\xi})$ for \mathbf{x}_{\min}^s in general i.e. to compute $f(\mathbf{x}_{\min}^s, \boldsymbol{\xi})$.

Till now, we have assumed that we have to rely only on already existing realizations (see IS above). There is the question: 'How to improve the IS solution?' Inspiration coming from stochastic programming suggests that we may derive new models by certain aggregation of original models to hedge against uncertainty that comes either from the fact that we do not know which model is right one or we want to be ready for all considered models. So, as one reasonable possibility how to improve the IS solution looks the idea to replace $\boldsymbol{\xi}$ with some compromise real values. It could be a convex combination or weighted average, and hence, precisely $E[\boldsymbol{\xi}]$, shortly $E\boldsymbol{\xi}$. However, this possibility is meaningless for models that are qualitatively too different with change of **a** or $\boldsymbol{\xi}$.

Definition 6 (EV deterministic reformulation) Let the UP (underlying program) be given. We define its here-and-now expected value (EV) deterministic reformulation:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ f(\mathbf{x}, E\boldsymbol{\xi}) \mid \mathbf{x} \in C(E\boldsymbol{\xi}) \} ,$$
 (8)

¹ In (2), we did not emphasize dependence of \mathbf{x} on \mathbf{a} because we use (2) in more general syntactical sense.

where $E\boldsymbol{\xi}$ denotes the expected value, z_{\min}^{EV} the minimal objective function value, and \mathbf{x}_{\min}^{EV} minimum.

The natural question appears : How good is the solution $\mathbf{x}_{\min}^{\rm EV}$ for the underlying objective function?

Definition 7 (EEV) For the EV deterministic reformulation, we define EEV:

$$EEV = E_{\boldsymbol{\xi}}[f(\mathbf{x}_{\min}^{EV}, \boldsymbol{\xi})] = E[\zeta^{EV}] .$$
(9)

We use the abbreviation EEV for the expected objective function value for the optimal solution of the Expected Value deterministic reformulation (the EEV abbreviation is traditionally used for this concept in stochastic programming literature.).

Using the previous paragraphs, we derive the idea of comparison of optimal solutions: 'Take the characteristic used for comparisons of ζ^{\odot} as an objective function.' Symbol ζ^{\odot} is used to denote the UP objective function value for any optimal solution of deterministic reformulation, even for this that will appear later. By this idea, we may choose $E[f(\mathbf{x}, \boldsymbol{\xi})]$ when we want to find a solution that is good in average for all considered models. In dependence on the application needs, further characteristics of ζ^{\odot} random variables could be utilized, e.g., modes, medians, quantiles, and sums of the expected value and multiple of standard deviation.

The EEV characteristic can be used to measure whether z_{\min}^{EV} looks realistic by computing the difference between the optimistic forecasted objective function value z_{\min}^{EV} and true average cost computed by EEV:

$$\mathrm{EEV} - z_{\mathrm{min}}^{\mathrm{EV}}$$
, for the maximum case: $z_{\mathrm{max}}^{\mathrm{EV}} - \mathrm{EEV}$.

So, we have found the way, how the optimal solutions of the different deterministic reformulations can be compared. It must be also emphasized that it would be too optimistic to make any conclusions about suitability of programs (deterministic reformulations) from conclusions made only about the optimal solutions of some instances of programs. Using the previous paragraphs, we derive the idea: 'Take the characteristic used for comparisons of ζ^{\odot} as an objective function.' By this idea, we choose $E[f(\mathbf{x}, \boldsymbol{\xi})]$.

Definition 8 (EO deterministic reformulation) Let the UP be given. We define its here-andnow expected objective (EO) deterministic reformulation:

$$? \in \underset{\mathbf{x}}{\operatorname{argmin}} \{ E[f(\mathbf{x}, \boldsymbol{\xi})] \mid \mathbf{x} \in \mathbb{R}^n \} .$$
(10)

We denote the minimal objective function value as z_{\min}^{EO} and minimum as $\mathbf{x}_{\min}^{\text{EO}}$. As before, for comparisons, we introduce ζ^{EO} i.e. $\zeta^{\text{EO}} = f(\mathbf{x}_{\min}^{\text{EO}}, \boldsymbol{\xi})$.

In general, the question is about the difference between $E[f(\mathbf{x}, \boldsymbol{\xi})]$ and $f(\mathbf{x}, E\boldsymbol{\xi})$. After the discussion about the optimal solutions, we may learn more about the relation of values of objective functions, and hence, about the difference $E[f(\mathbf{x}, \boldsymbol{\xi})] - f(\mathbf{x}, E\boldsymbol{\xi})$. The conclusion may be obtained by using known Jensen's inequality (1906) saying that for $f(\mathbf{x}, \boldsymbol{\xi})$ convex at $\boldsymbol{\xi}, E[f(\mathbf{x}, \boldsymbol{\xi})] \geq f(\mathbf{x}, E\boldsymbol{\xi})$.

At the end, we have to say something about the relation between EEV and the optimal EO objective function value. We know that $z_{\min}^{EO} = E[f(\mathbf{x}_{\min}, \boldsymbol{\xi})] = \text{globmin}_{\mathbf{x}} \{ E[f(\mathbf{x}, \boldsymbol{\xi})] \mid$

 $\mathbf{x} \in \mathbb{R}^n$ in general. As it is a global minimum, then $\forall \mathbf{x} \in \mathbb{R}^n : E[f(\mathbf{x}_{\min}^{EO}, \boldsymbol{\xi})] \leq E[f(\mathbf{x}, \boldsymbol{\xi})]$ in general. Then, specifically:

$$E[\zeta^{\text{EO}}] = E[f(\mathbf{x}_{\min}^{\text{EO}}, \boldsymbol{\xi})] \le E[f(\mathbf{x}_{\min}^{\text{EV}}, \boldsymbol{\xi})] = E[\zeta^{\text{EV}}] = \text{EEV} .$$

We have to note that those and previous inequalities require the existence of $E\boldsymbol{\xi}$ and $E[f(\mathbf{x},\boldsymbol{\xi})]$.

Then, we define the Value of Stochastic Solution (the VSS abbreviation is traditionally used in stochastic programming literature for the case.) as: $VSS = EEV - z_{min}^{EO}$. The concept is introduced to compare the here-and-now EO optimal solution and EV optimal solution by expectations of objective function values. The VSS measures how much can be saved when the true HN approach is used instead of the EV approach. A small value of the VSS measures that the approximation of the stochastic program by the EV program is a good one.

We have introduced the way how to compare optimal solutions of two here-and-now deterministic reformulations EV and EO by means of the VSS. In the similar way, we may find how to compare optimal solutions of WS and EO (HN) programs. We have used ζ^{\odot} only for HN programs. However, there is no restriction that does not allow us to use it also for the WS programs. Although $\mathbf{x}_{\min}(\boldsymbol{\xi})$ value changes randomly, it still can be applied to computation of the values of true objective function $f(\mathbf{x}, \boldsymbol{\xi})$ as before. We define a random variable $\zeta^{WS} = z_{\min}^{WS}(\boldsymbol{\xi}) = f(\mathbf{x}_{\min}^{WS}(\boldsymbol{\xi}), \boldsymbol{\xi})$. We define the Expected Value of Perfect Information (the EVPI abbreviation is traditionally used in stochastic programming literature and in stochastic models of operations research as well.) as: EVPI = $z_{\min}^{EO} - E[z_{\min}^{WS}(\boldsymbol{\xi})]$. The concept is introduced to compare the here-and-now EO optimal solution and WS optimal solution by expectations of objective function values. The EVPI measures how much it is reasonable to pay to obtain perfect information about the future. A small value of EVPI informs about little savings when we reach perfect information; the large EVPI says that the information about the future is valuable. If we obtain only sample information, the improvement on the optimum value is called the expected value of sample information (EVSI).

4. Risk Averse Objective Function Reformulations

Till now, we have used the expected value of $f(\mathbf{x}, \boldsymbol{\xi})$ as a good criterion to compare and find optimal solutions. The basic idea was to minimize 'average costs'. The idea is realistic when we have the opportunity to apply such a policy many times in the future. However, the average costs do not guarantee that there are no outlying costs. Therefore, we may think about some other criteria that are more 'risk averse'.

Definition 9 (VO deterministic reformulation) Let the UP be given. We define its here-andnow variance objective (VO) deterministic reformulation:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ \operatorname{var}[f(\mathbf{x}, \boldsymbol{\xi})] \mid \mathbf{x} \in \mathbb{R}^n \} .$$
 (11)

We denote the minimal objective function value as z_{\min}^{VO} and minimum as $\mathbf{x}_{\min}^{\text{VO}}$. As before, for comparisons, we introduce ζ^{VO} i.e. $\zeta^{\text{VO}} = f(\mathbf{x}_{\min}^{\text{VO}}, \boldsymbol{\xi})$.

In dependence on the application problem needs, further characteristics of ζ^{\odot} random variables in addition to E and var can be utilized. For example, we may choose the solution that behaves better for another than its own model i.e. we have introduced the VO program

to avoid large fluctuations of $f(\mathbf{x}, \boldsymbol{\xi})$. We can be even more strict and we can decide to minimize the maximum of fluctuations, therefore, minimize $\max_{\boldsymbol{\xi}} \{f(\mathbf{x}, \boldsymbol{\xi})\}$. We introduce:

Definition 10 (MM deterministic reformulation) Let the UP be given. We define its hereand-now min-max (MM) deterministic reformulation:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ \max_{\boldsymbol{\xi}^s} \{ f(\mathbf{x}, \boldsymbol{\xi}^s) \mid \boldsymbol{\xi}^s \in \Xi \} \mid \mathbf{x} \in \mathbb{R}^n \} .$$
 (12)

We denote the minimal objective function value as z_{\min}^{MM} and minimum as \mathbf{x}_{\min}^{MM} . As before, for comparisons, we introduce ζ^{MM} i.e. $\zeta^{MM} = f(\mathbf{x}_{\min}^{MM}, \boldsymbol{\xi})$.

We have seen various optimization criteria for HN deterministic reformulations (and hence for comparison of number of models). The question is what to do if we would like to follow several criteria at once. In fact, it is the same question as we have solved till now, however, on the higher level of abstraction. Previously, we have discussed the problem how to decide in front of future realizations $\boldsymbol{\xi}^s$. With discrete and finite probability distribution of $\boldsymbol{\xi}$, we could interpret it as a multicriteria optimization problem with many objective functions $f(\mathbf{x}, \boldsymbol{\xi}^s), \, \boldsymbol{\xi}^s \in \Xi$ and their importance is weighted by $p_s = P(\boldsymbol{\xi} = \boldsymbol{\xi}^s)$. In such a situation, our EO deterministic reformulation is equivalent to the choice of scalar reformulation based on the weighted average in multicriteria optimization. For some of other deterministic reformulations, we may find also its counterparts among scalar reformulations in multicriteria optimization. Underlying multicriteria optimization program (see [39]) is usually specified as ? $\in \operatorname{argmin} \{ \mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in C \}$, where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a vector objective function. Again the description is syntactically correct, however, without semantics as for the UP. Therefore, to solve the problem, at first, we have to specify a scalar reformulation. It is often based on some aggregation of \mathbf{f} components as, e.g., weighted average again. Applying the idea on the higher level, we may try to think how to aggregate, e.g., EO and VO objective functions in the same program: ? \in argmin $\{(E[f(\mathbf{x}, \boldsymbol{\xi})], \operatorname{var}[f(\mathbf{x}, \boldsymbol{\xi})]) \mid \mathbf{x} \in \mathbb{R}^n\}$. We may find the following formulation often used: $\min\{\lambda E[f(\mathbf{x},\boldsymbol{\xi})] + (1-\lambda) \operatorname{var}[f(\mathbf{x},\boldsymbol{\xi})] \mid \mathbf{x} \in \mathbb{R}^n\}$, where $\lambda \in [0,1]$ is a weighting parameter chosen by the wish of the modeler to prefer either low costs or small fluctuations. In addition to the choice of convex combination, we have to mention that aggregation may be formulated in more general way. The utility function can also unify the scales for different models.

As we have seen, different solutions (IS) can be compared also by building new derived models (EV, EO, VO, MM) that could be further combined to increase robustness of the optimal solution implemented. As we want to know how good is the optimal solution obtained for one model for other considered models then the question becomes : 'How the first model solution behaves in the second model?' When various models are considered and all their optimal solutions are collected, there is a question how any obtained solution can be evaluated for any listed model. The realization is quite simple. Following the ideas of EEV, VSS, and EVPI, we may compute ζ^{\odot} and then its necessary characteristic. We may further denote, e.g., $E[f(\mathbf{x}_{\min}^{EV}, \boldsymbol{\xi})]$ as \mathbf{x}_{\min}^{EV} at f^{EO} or even shortly: EV@EO, see [33] for further details.

We may extend and apply previous ideas also to the cases with probability and quantile objective functions introduced now. In applications, you may find various requirements, e.g., to increase reliability of some design. Therefore, we discuss how to optimize probability that is how to minimize probability of high costs and maximize probability of low costs. **Definition 11 (PO deterministic reformulation)** Let the UP (see Definition 3 for details) be given: $? \in \operatorname{argmin}_{\mathbf{x}} \{ f(\mathbf{x}, \boldsymbol{\xi}) \mid \mathbf{x} \in \mathbb{R}^n \}$. We define its here-and-now probabilistic objective (PO) deterministic reformulation (PO program):

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ P(f(\mathbf{x}, \boldsymbol{\xi}) > b) \mid \mathbf{x} \in \mathbb{R}^n \} ,$$
 (13)

where $b \in \mathbb{R}$ is a certain upper bound for the optimal objective function value (costs) that we do not want to exceed. We denote the minimal objective function value as z_{\min}^{PO} and the minimum as \mathbf{x}_{\min}^{PO} . As before, for comparisons, we introduce ζ^{PO} i.e. $\zeta^{PO} = f(\mathbf{x}_{\min}^{PO}, \boldsymbol{\xi})$. Notice please that the PO program (13) is the unconstrained one.

Theorem 1 (PO discrete case) We have the PO program with a discrete finite distribution of $\boldsymbol{\xi}$:

?
$$\in \underset{\mathbf{x}}{\operatorname{argmin}} \{ P(f(\mathbf{x}, \boldsymbol{\xi}) > b) \mid \mathbf{x} \in \mathbb{R}^n \} ,$$
 (14)

and $\forall \mathbf{x} \in \mathbb{R}^n$ we denote $P(f(\mathbf{x}, \boldsymbol{\xi}) = f(\mathbf{x}, \boldsymbol{\xi}^s)) = p(\boldsymbol{\xi}^s) = p_s$. We further assume that $\forall \mathbf{x} \in \mathbb{R}^n, \forall \boldsymbol{\xi}^s \in \Xi : f(\mathbf{x}, \boldsymbol{\xi}^s) - b$ is bounded from above by M. Then, the following 0-1 mathematical program solves the PO program:

$$? \in \underset{z, \mathbf{x}}{\operatorname{argmin}} \left\{ z \mid \forall s \in \mathcal{S} : \delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \leq b, \\ \sum_{s \in \mathcal{S}} p_s \left(1 - \delta_s \right) = z, \, \delta_s \in \{0, 1\}, \, s \in \mathcal{S} \right\} \,,$$

$$(15)$$

where S is a set of indices of realizations from Ξ . In addition, the constraint:

$$\delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \le b$$
 can be replaced by $f(\mathbf{x}, \boldsymbol{\xi}^s) \le b + M(1 - \delta_s)$.

Corollary 1 (Maximizing probability) For maximization: $? \in \underset{\mathbf{x}}{\operatorname{argmax}} \{P(f(\mathbf{x}, \boldsymbol{\xi}) \leq b) \mid \mathbf{x} \in \mathbb{R}^n\}$, we write:

$$? \in \operatorname*{argmax}_{z, \mathbf{x}} \left\{ z \mid \forall s \in \mathcal{S} : \delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \le b, \ \sum_{s \in \mathcal{S}} p_s \, \delta_s = z, \, \delta_s \in \{0, 1\}, \, s \in \mathcal{S} \right\}, \quad (16)$$

and we replace the implication constraints:

$$\delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \le b, \qquad f(\mathbf{x}, \boldsymbol{\xi}^s) \le b + M \left(1 - \delta_s\right).$$
(17)

The quantile in statistics is usually considered as a more robust characteristic in comparison with the mean. Therefore, it could be interesting to learn what we can expect when we minimize the quantile as a representative bound for the objective function level.

Definition 12 (QO deterministic reformulation) Let the UP (see Definition 3 for details) be given : $? \in \operatorname{argmin}_{\mathbf{x}} \{ f(\mathbf{x}, \boldsymbol{\xi}) \mid \mathbf{x} \in \mathbb{R}^n \}$. We define its here-and-now quantile objective (QO) deterministic reformulation (QO program):

$$? \in \underset{\mathbf{x}, z}{\operatorname{argmin}} \{ z \mid P(f(\mathbf{x}, \boldsymbol{\xi}) \le z) \ge \alpha \} , \qquad (18)$$

where $\alpha \in [0, 1]$ is a certain significance level for the objective function $f(\mathbf{x}, \boldsymbol{\xi})$ value that we want to have guaranteed. We denote the minimal objective function value as z_{\min}^{QO} and the minimum as \mathbf{x}_{\min}^{QO} . As before, for comparisons, we introduce ζ^{QO} i.e. $\zeta^{QO} = f(\mathbf{x}_{\min}^{QO}, \boldsymbol{\xi})$. Notice please that the QO program (18) is an unconstrained one.

Theorem 2 (QO discrete case) We have the QO program with a discrete finite distribution of $\boldsymbol{\xi}$:

$$? \in \underset{\mathbf{x}, z}{\operatorname{argmin}} \{ z \mid P(f(\mathbf{x}, \boldsymbol{\xi}) \le z) \ge \alpha \} , \qquad (19)$$

and $\forall \mathbf{x} \in \mathbb{R}^n$ we denote $P(f(\mathbf{x}, \boldsymbol{\xi}) = f(\mathbf{x}, \boldsymbol{\xi}^s)) = p(\boldsymbol{\xi}^s) = p_s$. We further assume that $\forall \mathbf{x} \in \mathbb{R}^n, \forall \boldsymbol{\xi}^s \in \Xi : f(\mathbf{x}, \boldsymbol{\xi}^s) - b$ is bounded from above by M.

Then, the following 0-1 mathematical program solves the QO program:

$$? \in \underset{z,\mathbf{x}}{\operatorname{argmin}} \left\{ z \mid \forall s \in \mathcal{S} : \delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \le z, \ \sum_{s \in \mathcal{S}} p_s \, \delta_s \ge \alpha, \ \delta_s \in \{0, 1\}, \ s \in \mathcal{S} \right\}, \quad (20)$$

where S is a set of indices of realizations from Ξ . In addition, the constraint:

$$\delta_s = 1 \Rightarrow f(\mathbf{x}, \boldsymbol{\xi}^s) \le z \quad \text{can be replaced by} \quad f(\mathbf{x}, \boldsymbol{\xi}^s) \le z + M (1 - \delta_s) .$$
 (21)

5. Conclusions

The typical engineering design applications have been listed and their requirements addressed to the optimization modeling have been emphasized. We have introduced basic concepts (MP, PMP, UP) in a unified way that allows a reader easily realize the shift from simplified deterministic cases to stochastic ones. Then, we have motivated and formally specified several deterministic reformulations. We have restricted ourselves to the objective functions in the first key step of the development of deterministic reformulations of applicable stochastic programs and also only to one-stage problems. We assume that these cases will be presented in detail and illustrated by engineering design examples in the subsequent papers. The presented reformulations are firstly linked to the traditional expected value (or individual scenario) cases that are suitable for the designs with less importance of risk evaluation. More advanced risk averse cases are also introduced and compared. For the important reliability linked PO model, discretization is introduced and a model is transformed into the one computable by modelling languages as GAMS. Several paragraph within the text have discussed the ideas of solution comparison from the original author's viewpoint.

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