TO THE PARAMETRIC ANTI-RESONANCE APPLICATION

Aleš Tondl*, Ladislav Půst**

A new application of the parametric anti-resonance is discussed. This phenomenon can be used not only for suppressing self-excited vibration or to reduce the externally excited vibration but also to suppress the parametric resonance when certain conditions are met. Another aim is to stimulate further investigations and the practical applications.

Keywords: *parametric excitation, parametric resonance, parametric anti-resonance, additional parametric excitation*

1. Introduction

Let us consider a parametrically excited system which, after transformation into the quasi-normal form, can be governed by the following equations:

$$\ddot{x}_s + \Omega_s^2\, x_s + \varepsilon \left[\sum_{k=1}^{n} (\Theta_{sk}\, \dot{x}_k + Q_{sk}\, x_k) \right] = 0 \ , \quad (s = 1, 2, \dots, n) \ , \qquad (1)$$

where ε is small parameter, Ω_s are the natural frequencies of the abbreviated system and Θ_{sk}, Q_{sk} are periodic functions of the parametric excitation with frequency ω.

Note: Quite similar equations can be obtained when analysing the stability of the externally excited vibration in the interval of the excitation frequency where due to the nonlinearity a nonlinear resonance like the subharmonic resonance can occur.

It is well known that the system can be unstable due to the action of the parametric excitation. This can destabilize the system in certain intervals of frequency ω lying at

$$\omega = \frac{\Omega_j \pm \Omega_k}{N} \ , \quad (N = 1, 2, \dots) \ . \qquad (2)$$

For $k = j$ and for plus sign we speak about the instability interval of the first kind and Nth order. For $k \neq j$ the combination parametric resonance can be initiated. Due to the nonlinear progresive damping in the instability intervals limited steady vibrations are initiated. In [1] the method for determination of these instability intervals was presented and as an example it was proved the following result:

In the case of parametric excitation only due to the stiffness periodic variation only for one sign in (2) the instability interval can exist (e.g. for $\omega \cong |\Omega_j + \Omega_k|$). There is a question what effect occurs for the other sign (e.g. for $\omega \cong |\Omega_j - \Omega_k|$). It was discovered (see [2], [3]) that in this case the parametric excitation has a stabilization effect resulting

* Ing. A. Tondl, DrSc., Zborovská 41, 15000 Praha 5
** Ing. L. Půst, DrSc., Institute of Thermomechanics AS CR, v.v.i., Dolejškova 5, 18200 Praha 8

e.g. in suppressing self-excited vibration. There exist a numerous literature dealing with this problem where for different systems the influence on self-excited and externally excited vibration have been analyzed (see especially [4], the references survey is in [5]). It was proved that the self-excited vibration can be even fully suppressed, which is not the case as for externally excited vibrations (there only the vibration limiting can be achieved).

In this contribution the attention will be paid to the following question: Is it possible to suppress the parametric resonance by an additional parametric excitation?

2. Basic analysis

First of all let us suppose that both parametric excitations (original and the additional) are harmonic. Such a system after transformation into the quasi-normal form is governed by the following equations:

$$\ddot{x}_s + \Omega_s^2 x_s + \varepsilon \left\{ \sum_{k=1}^{n} [\Theta_{sk}\,\dot{x}_k + \cos\omega t \cdot Q_{sk}\,x_k + \cos\eta t \cdot P_{sk}\,x_k] \right\} = 0\,, \quad (s = 1, 2, \ldots, n)\,, \tag{3}$$

where ε is a small parameter, ω is the frequency of the acting original parametric excitation and η is the frequency of the additional parametric excitation which should suppress the parametric resonance of the original parametric excitation and P_{sk}, Q_{sk} are the coefficients.

Let us suppose that the aim of the additional parametric excitation is to suppress the parametric resonance of the first kind and first order, e.g. at $\omega = 2\,\Omega_1$.

There is necessary to take into account the following facts:

1) Considering the case $\cos\eta t = 0$ the resonance at $\omega = 2\,\Omega_1$ is the parametric resonance of the first kind. The trivial solution is unstable, unless the following condition is met (see [1]):

$$\left(\frac{Q_{11}}{2\,\Omega_1}\right)^2 - \Theta_{11}^2 \geq 0\ . \tag{4}$$

For positive damping is Θ_{11} positive and so the condition for suppression this parametric resonance reads:

$$\frac{Q_{11}}{2\,\Omega_1} \leq \Theta_{11}\ . \tag{5}$$

2) Considering the case when $\cos\omega t = 0$, $\cos\eta t = \cos(|\Omega_k - \Omega_1|)t$, then the conditions to eliminate the effect of the negative linear damping are (see [1] and Appendix):

$$\Theta_{11} + \Theta_{kk} \geq 0\,, \qquad \frac{Q_{1k}\,Q_{k1}}{4\,\Omega_1\,\Omega_k} + \Theta_{11}\,\Theta_{kk} \geq 0\ . \tag{6}$$

For positive damping the first condition (6) is met and the second is decisive. We can see that the first term in the second condition (6) can represent the additional positive damping when $Q_{1k}\,Q_{k1}$ is a positive value. For the positive linear damping the conditions for avoiding the parametric resonance at $\omega = 2\,\Omega_1$ is:

$$\frac{Q_{1k}\,Q_{k1}}{4\,\Omega_1\,\Omega_k} - \left(\frac{Q_{11}}{2\,\Omega_1}\right)^2 + \Theta_{11}^2 + \Theta_{11}\,\Theta_{kk} \geq 0\ . \tag{7}$$

For positive damping the sufficient condition of the parametric resonance suppression at $\omega = 2\,\Omega_1$ reads:

$$\frac{Q_{1k}\,Q_{k1}}{\Omega_k} - \frac{Q_{11}^2}{\Omega_1} \geq 0 \; . \tag{8}$$

3. Example

Let us consider a two-mass system with masses m_1, m_2 resisting on springs with periodically variable stiffneses $k_1 = k_{10}\,(1 + \varepsilon\,\alpha_1\,\cos\varpi t)$, $k_2 = k_{20}\,(1 + \varepsilon\,\alpha_2\,\cos\chi t)$ (see Fig. 1).

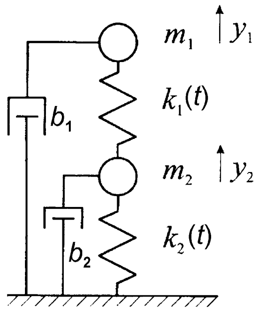

Fig.1: Schema of the system

The motion of both masses is positively damped. The damping of mass m_1 consists of a linear viscous component and a progressive component. The latter component is convenient for numerical solution in order to obtain a limited vibration in case of unstable equilibrium position; for stability of the equilibrium position can be left out. For mass m_2 only linear damping is considered.

Denoting the deflections of masses as y_1, y_2 the system is governed by the following equations:

$$\begin{aligned} & m_1\,\ddot{y}_1 + k_{10}\,(y_1 - y_2) + \varepsilon\,\big[(b_1 + d\,y_1^2)\,\dot{y}_1 + (k_{10}\,\alpha_1\,\cos\varpi t)\cdot(y_1 - y_2)\big] = 0 \; . \\ & m_2\,\ddot{y}_2 - k_{10}\,(y_1 - y_2) + k_{20}\,y_2 + \\ & \qquad + \varepsilon\big[b_2\dot{y}_2 - k_{10}\,\alpha_1\,(y_1 - y_2)\,\cos\varpi t + k_{20}\,\alpha_2\,y_2\,\cos\chi t\big] = 0 \; . \end{aligned} \tag{9}$$

where m_1, m_2 are the masses, k_{10}, k_{20} are the average stiffnesses of the springs, b_1, b_2 are linear damping coefficients, d is the coefficient of the progressive damping, ϖ frequency of the harmonic parametric excitation and χ is the frequency of the additional harmonic parametric excitation.

After rearranging and time transformation ($\omega_1\,t = \tau$, $\omega_1 = \sqrt{k_{10}/m_1}$) the following equations are obtained:

$$\begin{aligned} y_1'' + y_1 - y_2 + \varepsilon\,\big[(\kappa_1 + \delta\,y_1^2)\,y_1' + \alpha_1\,(y_1 - y_2)\,\cos\omega\tau\big] = 0 \; , \\ y_2'' - M\,(y_1 - y_2) + q^2\,y_2 + \varepsilon\,\big[\kappa_2\,y_2' - M\,\alpha_1\,(y_1 - y_2)\,\cos\omega\tau + q^2\,\alpha_2\,y_2\,\cos\eta\tau\big] = 0 \; , \end{aligned} \tag{10}$$

where

$$M = \frac{m_1}{m_2} \; , \quad q^2 = \frac{\dfrac{k_{20}}{m_2}}{\dfrac{k_{10}}{m_1}} \; , \quad \frac{k_{10}}{m_1} = \omega_1^2 \; , \quad \kappa_1 = \frac{b_1}{m_1\,\omega_1} \; ,$$

$$\delta = \frac{d}{m_1\,\omega_1} \; , \quad \kappa_2 = \frac{b_2}{m_2\,\omega_1} \; , \quad \omega = \frac{\varpi}{\omega_1} \; , \quad \eta = \frac{\chi}{\omega_1} \; .$$

Equations (10) can be transformed into the quasi-normal form using equations:

$$y_1 = x_1 + x_2 \ , \qquad y_2 = a_1\, x_1 + a_2\, x_2 \tag{11}$$

and equations (10) get the form:

$$\begin{aligned} x_1'' + \Omega_1^2\, x_1 + \varepsilon\, F_1 = 0 \ , \\ x_2'' + \Omega_2^2\, x_2 + \varepsilon\, F_2 = 0 \ , \end{aligned} \tag{12}$$

where

$$\begin{aligned} &(\Omega^2)_{1,2} = \frac{1}{2}\left(1 + M + q^2\right) \mp \left[\frac{1}{4}\left(1 + M + q^2\right)^2 - q^2\right]^{1/2} \ , \\ &F_1 = \frac{1}{a_1 - a_2}\left(-a_2\, \Phi_1 + \Phi_2\right) \ , \qquad F_2 = \frac{1}{a_1 - a_2}\left(a_1\, \Phi_1 - \Phi_2\right) \ , \\ &\Phi_1 = \kappa_1\left(x_1' + x_2'\right) + \alpha_1\left[(1 - a_1)\, x_1 + (1 - a_2)\, x_2\right] \cos\omega\tau \ , \\ &\Phi_2 = \kappa_2\left(a_1\, x_1' + a_2\, x_2'\right) - M\, \alpha_1\left[(1 - a_1)\, x_1 + (1 - a_2)\, x_2\right] \cos\omega\tau + \\ &\qquad\qquad + q^2\, \alpha_2\left(a_1\, x_1 + a_2\, x_2\right) \cos\eta\tau \ . \end{aligned}$$

Now let us investigate the conditions for suppressing the parametric resonance at $\omega = 2\,\Omega_1$ when $\eta = \Omega_2 - \Omega_1$. For simplicity let us consider that no linear damping exists, i.e. $\kappa_1 = \kappa_2 = 0$. Then for full suppression of the parametric resonance the following condition (see (7)) must be met:

$$\frac{Q_{12}\, Q_{21}}{\Omega_1\, \Omega_2} - \left(\frac{Q_{11}}{\Omega_1}\right)^2 \geq 0 \ , \tag{13}$$

where

$$Q_{12}\, Q_{21} = -\frac{\alpha_2^2\, a_1\, a_2\, q^4}{(a_1 - a_2)^2} \ , \qquad Q_{11} = -\frac{\alpha_1\, a_2\, (1 - a_1)}{a_1 - a_2} \ .$$

Considering that the following relations are met (see [6]):

$$\begin{aligned} &0 \leq a_1 \leq 1 \ , \qquad -a_2 \geq 0 \ , \qquad a_1\, a_2 = -M \ , \\ &a_{1,2} = -\frac{1}{2}\left(M + q^2 - 1\right) \pm \left[\frac{1}{4}\left(M + q^2 - 1\right)^2 + M\right]^{1/2} \end{aligned}$$

the condition (13) can be simplified as follows:

$$\alpha_2 \geq \alpha_1\, \frac{(1 - a_1)^2\, (-a_2)}{q^2 \sqrt{M\, \dfrac{\Omega_1}{\Omega_2}}} = \alpha_1\, \Lambda_1 \ . \tag{14}$$

In the similar way it can be derived that for suppressing of the parametric resonance at $\omega = 2\,\Omega_2$ (also for $\kappa_1 = \kappa_2 = 0$) the following condition should be fulfilled:

$$\alpha_2 \geq \alpha_1\, \Lambda_2 \ ,$$

where

$$\Lambda_2 = \frac{(1 - a_2)^2\, a_1}{q^2 \sqrt{M\, \dfrac{\Omega_2}{\Omega_1}}} \ . \tag{15}$$

In Fig. 2 $\Lambda_1(q)$ is presented for several values of M and similarly $\Lambda_2(q)$ in Fig. 3.

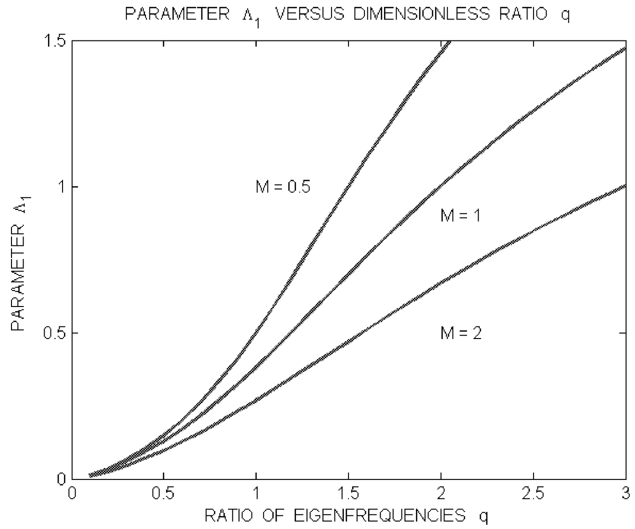

Fig.2: Parameter Λ_1 for suppressing parametric resonance at $\omega = 2\,\Omega_1$

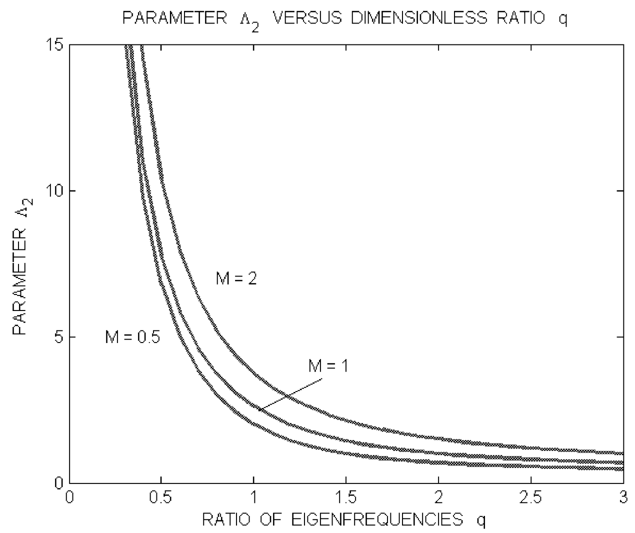

Fig.3: Parameter Λ_2 for suppressing parametric resonance at $\omega = 2\,\Omega_2$

Comparing the course of $\Lambda_1(q)$ and $\Lambda_2(q)$ we can see that $\Lambda_1(q)$ is an increasing function with increasing q while the opposite is true concerning $\Lambda_2(q)$, which is a decreasing function. It should be noticed that Λ_2 is significantly higher than Λ_1. Consequently to suppress the parametric resonance at $\omega \cong 2\,\Omega_1$ is much easier than the parametric resonance at $\omega \cong 2\,\Omega_2$. Even the optimal value of q is different. The diagrams of $\Lambda_1(q)$, $\Lambda_2(q)$ help to select the convenient system tuning considering the possibilities and to decide whether only one parametric resonance (e.g. at $\omega \cong 2\,\Omega_1$) or both parametric resonances (also at $\omega \cong 2\,\Omega_2$) should be suppressed. For example for $M = 1$, $q = 1$ is $\Lambda_1 \cong 0.38$ but $\Lambda_2 \cong 2.6$, i.e. the first resonance can be suppressed provided the additional parametric excitation with frequency $\eta = \Omega_2 - \Omega_1$ with the amplitude nearly one third of the amplitude of the parametric excitation with frequency ω. Approximately the relation $\Lambda_1 = \Lambda_2$ is valid for $M = 1$, $q = 2$.

4. Numerical examples

The numerical solution of selected cases is used for verification of analytically gained result.

Time history of motions y_1, y_2 of both masses is solved by Runge-Kutta procedure according to the equations (10). System tuning was selected in the centers of the first parametric instability region, which corresponds following values: $M = 1$, $q = 1$, $\varepsilon = 0.1$, $\kappa_1 = 0$, $\kappa_2 = 0$, $\omega = 1.236$. Limitation of amplitudes in this instability domain is reached by nonlinear progressive damping coefficient $\delta = 0.001$. Very small appr. ($10^{-5}\, x_{\max}$) but not zero initial conditions were applied to excite the increase of displacements y_1, y_2 during acceptable time of transient processes.

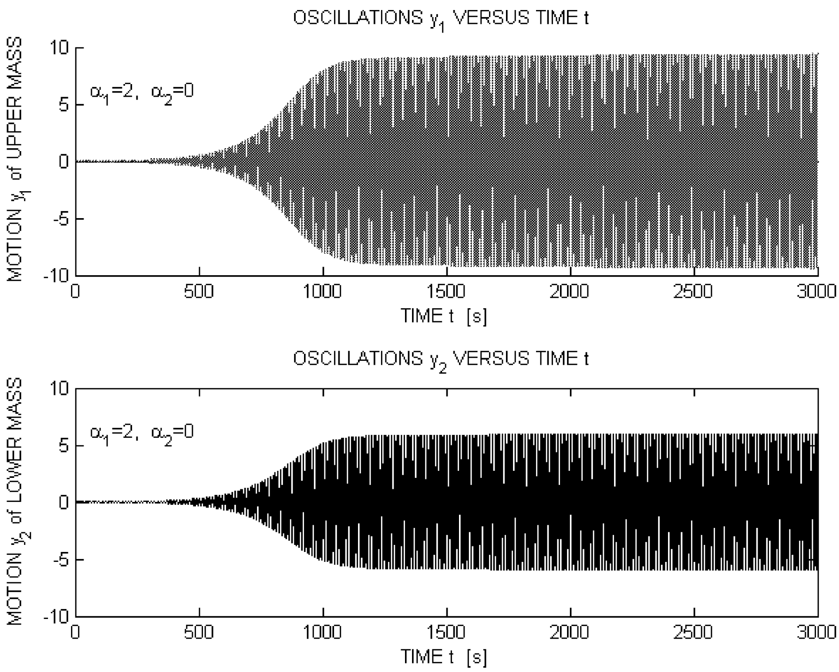

Fig.4: Increase of oscillations at only one parametric excitation – $\alpha_1 = 2$, $\alpha_2 = 0$

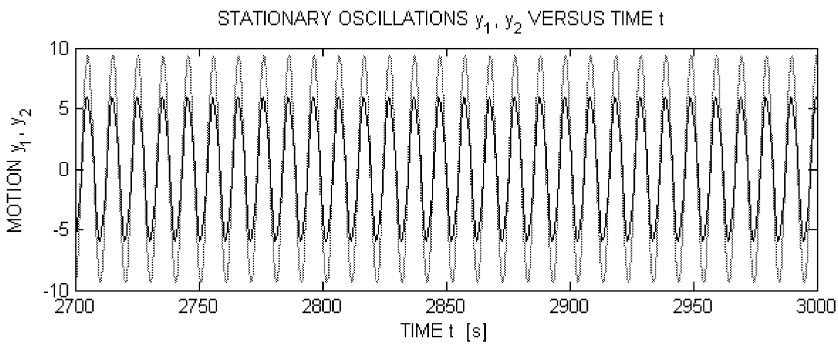

Fig.5: Form of parametric oscillations in stationary state – $\alpha_1 = 2$, $\alpha_2 = 0$

The evolution of parametric oscillations and following stabilization of $y_1(t)$, $y_2(t)$ up to $t = 3000$ is shown in Fig. 4 for only one parametric excitation $\alpha_1 = 2$, $\alpha_2 = 0$.

In Fig. 5 is ZOOM view in time interval $t \in (2700, 3000)$, where the approximately harmonic form of oscillations $y_1(t)$, $y_2(t)$ is evident.

If on the parametrically excited system with frequency $\omega = 1.236$ acts also auxiliary parametric oscillation with frequency $\eta = \Omega_2 - \Omega_1 = 1$ and with small amplitude $\alpha_2 = 0.3$, the destabilizating effect of the first parametric excitation is partially suppressed. It is seen from Fig. 6, where the evolution of oscillation is delayed to time interval $t \in (1000, 1500)$. The consolidated motion is not yet harmonic, but contains small chaotic component as seen also in Fig. 7, where the ZOOM course is drawn in time interval $t \in (2700, 3000)$.

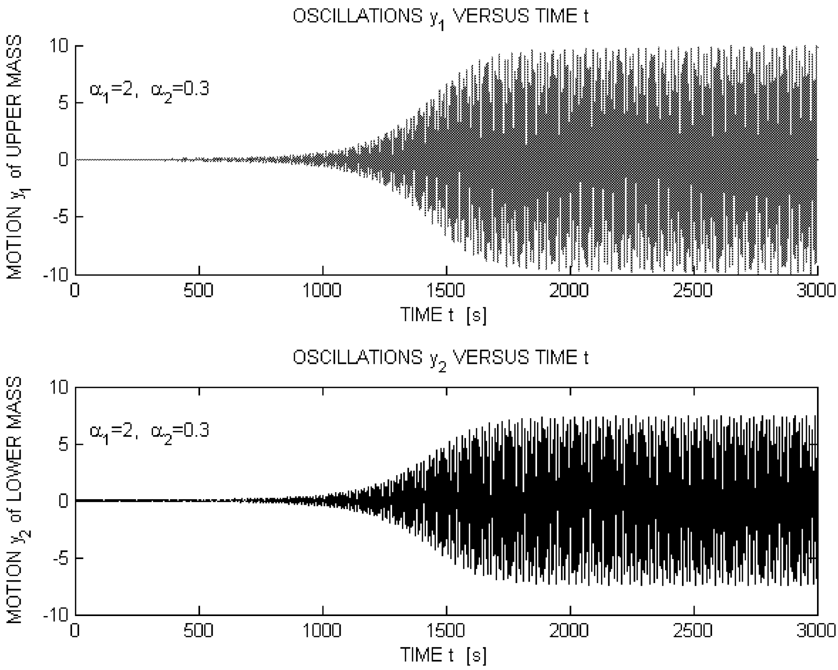

Fig.6: Increase and stabilization of oscillations at low auxiliary parametric excitation – $\alpha_1 = 2$, $\alpha_2 = 0.3$

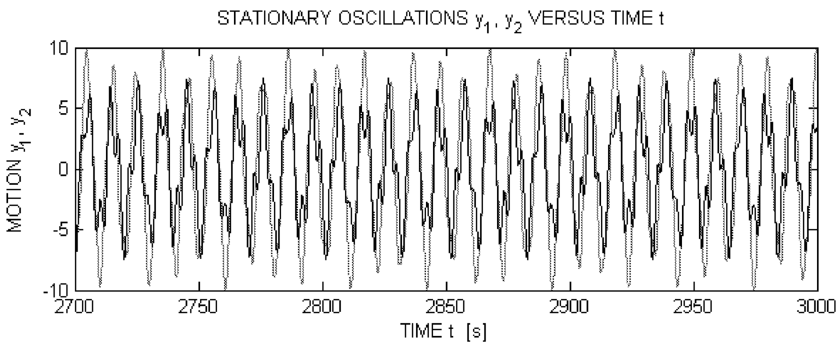

Fig.7: Non-harmonic oscillations at low auxiliary parametric excitation – $\alpha_1 = 2$, $\alpha_2 = 0.3$

Further increase of auxiliary excitation to $\alpha_2 = 0.7$ changes the evolution stage oscillations into quasi-beats motion, Fig. 8. Detail of such kind of oscillation is shown in Fig. 9, again in time interval $t \in (2700, 3000)$.

When the level of auxiliary parametric excitation is set up on $\alpha_2 = 1$ (at $\alpha_1 = 2$), the oscillations y_1, y_2 of both masses are completely suppressed, as shown in Fig. 10.

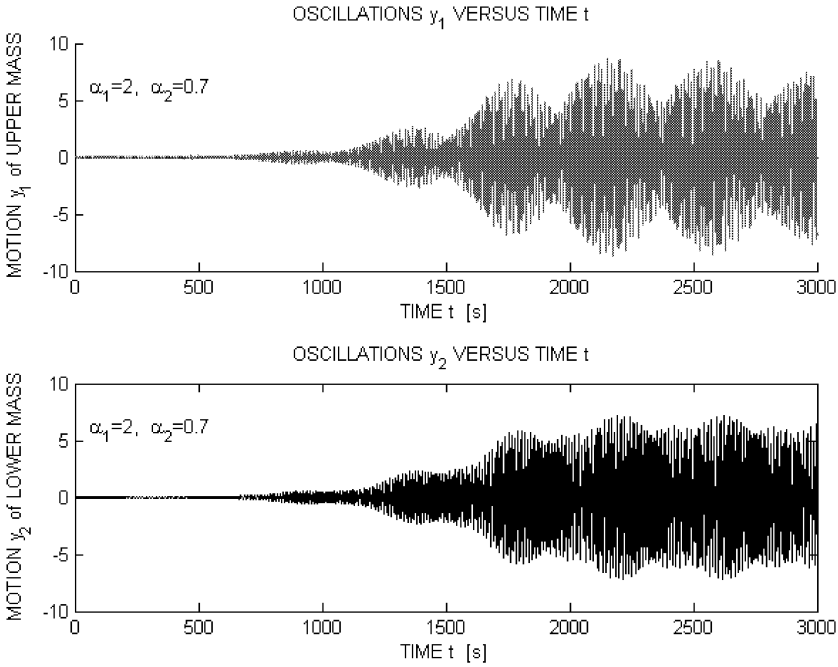

Fig.8: Transient and quasi-beats oscillations at higher auxiliary parametric excitation – $\alpha_1 = 2$, $\alpha_2 = 0.7$

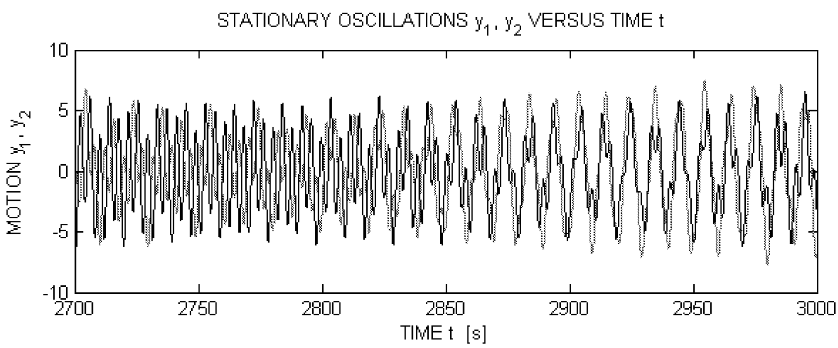

Fig.9: ZOOM time history of quasi-beat oscillations at $\alpha_1 = 2$, $\alpha_2 = 0.7$

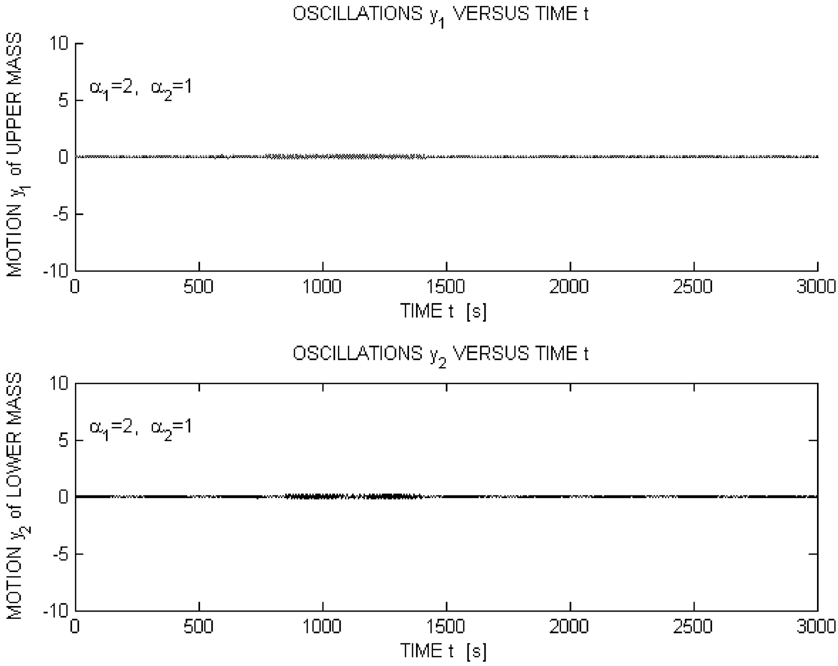

Fig.10: Fully suppressed oscillations at sufficient level of auxiliary parametric excitation – $\alpha_1 = 2$, $\alpha_2 = 1$

5. Conclusion

It was shown that the parametric excitation using the phenomenon of the parametric anti-resonance has a broader field of application for suppressing undesirables vibrations. Not only self-excited vibrations but also parametric resonance can be suppressed by an additional parametric excitation when certain conditions are met. The parametric excitation can also be used for the full suppressing of typically non-linear externally excited resonances like the subharmonic ones. Indeed when analyzing the stability of externally excited nonlinear system, the relevant differential equations including perturbations have periodic variable coefficients.

The presented analysis is the first step to further analytical as well as numerical analyses intended for the optimal tuning conditions development and their practical applications.

6. Appendix

Let us consider a system with a harmonic parametric excitation, which in quasi-normal form is governed by the following equations:

$$\ddot{x}_s + \Omega_s^2 \, x_s + \sum_{k=1}^{n} (\Theta_{sk} \, \dot{x}_k + Q_{sk} \, x_k \, \cos \omega t) = 0 \ , \quad (s, k = 1, 2, \ldots, n) \ . \tag{16}$$

To stabilize the system in the neighborhood of

$$\omega = \omega_0 = |\Omega_j - \Omega_k| \tag{17}$$

the following conditions must be met (see example in [1]) :

$$\Theta_{jj} + \Theta_{kk} \geq 0 \ ,$$
$$\frac{Q_{jk}\, Q_{kj}}{4\,\Omega_j\,\Omega_k} + \Theta_{jj}\,\Theta_{kk} \geq 0 \ . \tag{18}$$

Acknowledgement

The work was supported by the research project of the Czech Science Foundation No. 101/08/0282 'Mechatronické pohonové soustavy s nelineárními vazbami'.

References

[1] Tondl A.: Metoda k určení intervalů nestability quasi-harmonických kmitů (The method for the determination of instability intervals of quasi=harmonic vibration systems), Aplikace matematiky, 4, 1959, No. 4, 278–289
[2] Tondl A.: To the interaction of different types of excitation, Proc. Seminar Interaction and Feed Back '97, Institute of Thermomechanics, Czech Academy of Sciences, Prague 1997, 111–118
[3] Tondl A.: To the problem of quenching self-excited vibrations. Acta Technica ČSAV, 43 (1998), 109–116
[4] Ecker H.: Suppression of Self-Excited Vibrations in Mechanical Systems by Parametric Stiffness Excitation, Forschriftsberichte Simulation, Band 11, ARGESIM/ASIM-Verlag, Wien, 2003
[5] Tondl A.: To the problem of self-excited vibration suppression, Egineering Mechanics, Vol. 15, 2008, No. 4, p. 297–308
[6] Tondl A.: On the Interaction between Self-excited and Forced Vibrations, Monographs and Memoranda No. 20, National Research Institute for Machine Design, Běchovice, 1976

Received in editor's office: May 7, 2010
Approved for publishing: June 25, 2010