

## ROADWAY AUTOMOBILE STABILITY A NUMERICAL STUDY

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*A mathematical model of the roadway automobile motion is numerically analyzed. This model is intended to describe the roadway automobile stability. A previous paper [6] described the model in detail and the general method of qualitative analysis. In the present paper, we continue the discussion of stability by numerical simulations and the specific question we attempted to answer is: which parameter(s) of automobile geometry and quality of the roadway can serve as a reliable predictor(s) for car crash? Data from Daimler-Chrysler AG and Ford Motor Company Limited were used for that purpose, considering three car types – Mercedes-Benz E 320 (T-modelle), Ford Focus and Mercedes-Benz Sprinter (1). Hence, one can consider the present work as a natural continuation of [6].*

Key words: roadway automobile stability, numerical analysis, nonlinear system

### 1. Introduction

Mathematical models have been used to investigate the road way automobile motion [1–6], and optimal car velocity in traffic jam [7]. Those authors introduced the idea of modeling the automobile motion and automobile stability as simple as possible.

The model we have used for our study is depicted in Fig. 1, which shows the scheme of roadway of an automobile motion in the plane XOY. According to this scheme and [1], in a previous our paper [6] we examined qualitatively a modified 3×3 autonomous, nonlinear system of ordinary differential equations modeling the roadway automobile motion. This autonomous system (see [1] and [6] for a complete derivation) has the form

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= A_1 x_1 - A_2 x_2 - A_3 x_3 + A_4 x_1^3, \\ \dot{x}_3 &= A_6 x_1 - A_7 x_2 - A_5 x_3 + A_8 x_1^3.\end{aligned}\tag{1}$$

The variables  $x_1$  to  $x_3$  present dimensionless angles of deviation, angular velocity and cross velocity, respectively. The constant coefficients  $A_1$  to  $A_8$  are dimensionless algebraic complexes of the characteristic values of the automobile system and have the form

$$\begin{aligned}A_1 &= \frac{(\alpha_1 + 2 K_1 a_1 - 2 K_2 a_2) T^2}{I}, & A_2 &= \frac{2 (K_1 a_1^2 + K_2 a_2^2) T}{I V}, \\ A_3 &= \frac{2 (K_1 a_1 - K_2 a_2) T l_0}{I V \theta_0}, & A_4 &= \frac{\alpha_2 T^2 \theta_0^2}{I},\end{aligned}\tag{2}$$

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$$\begin{aligned}
 A_5 &= \frac{2(K_1 + K_2)T}{mV}, & A_6 &= \frac{[2(K_1 + K_2) + \chi_1]T^2\theta_0}{ml_0}, \\
 A_7 &= \frac{2(K_1 a_1 - K_2 a_2)T\theta_0}{IVl_0}, & A_8 &= \frac{\chi_2 T^2 \theta_0^3}{ml_0}.
 \end{aligned} \tag{2}$$

where  $\chi_1, \chi_2$  are constants having the dimension of force,  $\alpha_1, \alpha_2$  are constants with moment dimension,  $m$  is the mass of the automobile,  $V$  is the velocity of the automobile, and  $a_1, a_2, I, K_1$  and  $K_2$  are positive constants which has been introduced by Rocard in [1]. The constants  $\theta_0, T$  and  $l_0$  are the characteristic values of the angle deviation, time and cross velocity, respectively. In [6], we obtained that the system (1) has three equilibrium points if the relations

$$\begin{cases}
 A_4 A_5 > A_3 A_8 \\
 A_3 A_6 > A_1 A_5 \\
 A_3 \neq 0
 \end{cases} \tag{3}$$

or

$$\begin{cases}
 A_4 A_5 < A_3 A_8 \\
 A_3 A_6 < A_1 A_5 \\
 A_3 \neq 0
 \end{cases} \tag{4}$$

are valid. If case (3) or (4) are not valid, the equilibrium point is only one with values  $x_1^s = 0, x_2^s = 0, x_3^s = 0$ . The strategy that we adopt throughout this study is to use all analytical tools (obtained from us in [6]) for investigation of the stability of three car types, i.e., Mercedes-Benz E-320 (T-modelle); Ford Focus and Mercedes-Benz Sprinter (1). Basically, all we need for our purposes is the expression of the first Lyapunov value ( $L_1(\lambda_0)$ ) calculated on the boundary of stability  $R = 0$  in equation (20) of the previous paper [6] and its shown here in Appendix – Eq. (A.2). In this paper we present further numerical results of this preliminary analytical study. In the previous paper we looked at the conditions for stability and showed that a very general class of controller parameters would give stable solutions of the model (1). Here we examine (by numerical experiments) the stable and unstable states and show that the stability loss can be two types: ‘soft’ – reversible or ‘hard’ – irreversible.

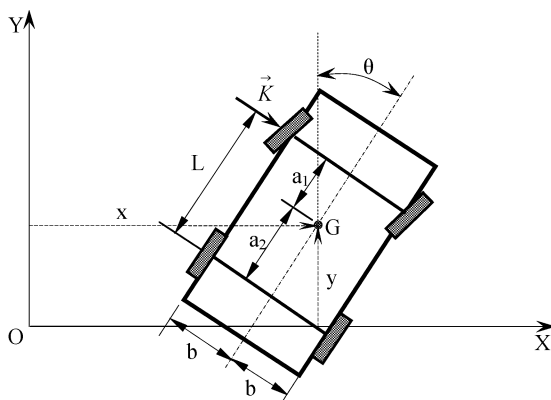


Fig.1: Simplified scheme of roadway of an automobile motion; with here we denote the external side force which turned away the automobile

## 2. Numerical experiments

In this section, we examine the mathematical model presented by the Eq.(1) for the dimensionless angles of deviation  $x_1$ , angular velocity  $x_2$  and cross velocity  $x_3$ .

As mentioned earlier, the automobile constants  $m$ ,  $a_1$ ,  $a_2$  were taken from [8–10]. Their values for Mercedes-Benz E-320 (T-modelle), Ford Focus and Mercedes-Benz Sprinter (1) are shown in Table 1. Following [1], the automobile inertance moment  $I$  was calculated by the equation

$$I = m \varrho^2, \quad (5)$$

where  $\varrho$  is the inertial radius. From the literature [2], we take the approximate values for the inertial radius  $\varrho = 1.2$ . According to [4, 11–13] for  $T$ ,  $\theta_0$  and  $l_0$  we take the average values:  $T = 1$  s,  $\theta_0 = 10$ ,  $l_0 = 0.1$  m. The different values of  $I$  (for the three car types) are also shown in Table 1. From [1], for the lateral climb, we can write

$$K = 1.5 \text{ mg}, \quad (6)$$

where  $K$  can be  $K_1$  or  $K_2$ .

	$m$ [kg]	Base [mm] ( $L = a_1 + a_2$ )	$I$ [N m s <sup>2</sup> ]	$K$ [N]
Mercedes-Benz E-320 (T-modelle)	2100	2833	3024	30870
Ford Focus	1500	2615	2160	22050
Mercedes-Benz Sprinter (1)	3700	3000	5328	54390

Tab.1: Values of  $m$ ,  $L = a_1 + a_2$ ,  $I$  and  $K$  for three make automobiles, i.e., Mercedes-Benz E-320 (T-modelle); Ford Focus and Mercedes-Benz Sprinter (1)

Let us first investigate the system (1), when the corresponding values for the Mercedes-Benz E-320 (T-modelle) must hold.

### (i) Mercedes-Benz E-320 (T-modelle)

In this case, the corresponding numerical values of the dimensionless parameters  $A_1$  to  $A_8$  can be calculated by substituting of the first row values (see Table 1) into (4). As a result, we obtain

$$\begin{aligned} A_1 &= -1.7585, & A_2 &= 2.86, & A_3 &= -0.0063, & A_4 &= 9.92 \times 10^{-4}, \\ A_5 &= 2.0525, & A_6 &= 572.667, & A_7 &= -0.9081, & A_8 &= -0.019, \end{aligned} \quad (7)$$

where  $a_1 = a_2 = 1.4165$  m,  $K_2 = 30870$  N,  $V = 27.78$  m/s. Because the mass of this make automobile noted in Table 1 is averaged one, we choose  $K = K_1 = 29000$  N. Here we note that  $K_1 a_1 - K_2 a_2 < 0$ . In view of the lack of data for parameters  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$  we assume to vary the parameters  $A_1$ ,  $A_4$ ,  $A_6$  and  $A_8$ . Also, we vary the automobile velocity  $V$ , when the parameters  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$  are fixed. All numerical realizations of the system (1) are accomplished at initial conditions  $x_1 = 0.1$ ,  $x_2 = 0$ ,  $x_3 = 0$ . Certainly, the equilibrium state (fixed point) of the system is always different from these conditions.

In Figure 2, the curves of the angles of deviation  $x_1$ , the angular velocity  $x_2$  and the cross velocity  $x_3$  are shown. After 3 or 4 seconds, the angular velocity  $x_2$  decreases to zero, and

the angles deviation  $x_1$  and the cross velocity  $x_3$  increase to  $2.8 \times 10^{-3}$  and 0.75, respectively. In this case, the Routh-Hurwitz conditions for stability

$$p = A_2 + A_5 > 0, \quad (8)$$

$$q = A_2 A_5 - A_1 - A_3 A_7 > 0, \quad (9)$$

$$r = A_3 A_6 - A_1 A_5 > 0, \quad (10)$$

$$R = p q - r > 0 \quad (11)$$

are valid, i.e. the steady state  $x_1^s = 0$ ,  $x_2^s = 0$ ,  $x_3^s = 0$  of the system (1) is stable. Here we note that the conditions (3) or (4) in this case are not valid and steady state of (1) is only one.

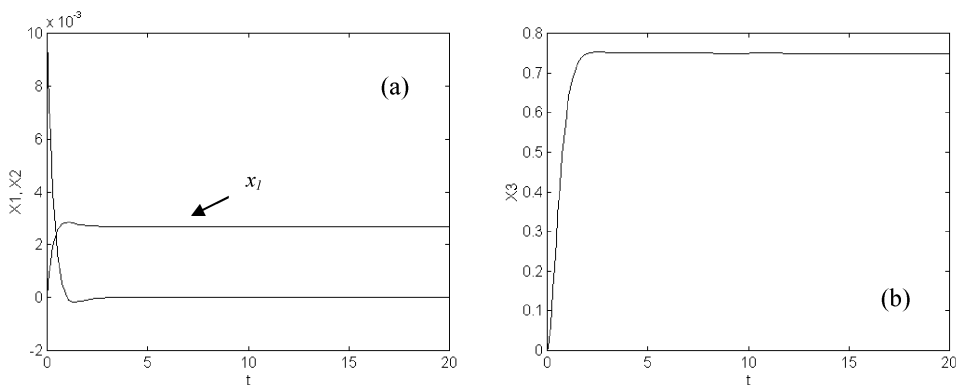


Fig.2: Stable solutions for  $x_1$ ,  $x_2$  (a) and  $x_3$  (b) at  $A_1 = -1.7585$ ,  $A_2 = 2.86$ ,  $A_3 = -0.0063$ ,  $A_4 = 9.92 \times 10^{-4}$ ,  $A_5 = 2.0525$ ,  $A_6 = 572.667$ ,  $A_7 = -0.9081$ ,  $A_8 = -0.019$ ; here, the automobile velocity is  $V = 27.78$  m/s, i.e. 100 km/h

In Figure 3, we illustrate the results of the computations for  $A_1 = -1.7585$ ,  $A_2 = 1.43$ ,  $A_3 = -0.00315$ ,  $A_4 = 9.92 \times 10^{-4}$ ,  $A_5 = 1.03$ ,  $A_6 = 572.667$ ,  $A_7 = -0.454$  and  $A_8 = -0.019$ . In this case the automobile velocity is  $V = 55.56$  m/s, i.e. 200 km/h, and the Routh-Hurwitz condition for stability (Eq. (11)) is equal to 7.94. Here, we see that for these values of parameters  $A_1$  to  $A_8$  the system (1) has stable solutions, too. As  $V$  is increased to 83.34 m/s, i.e. 300 km/h, there are stable solutions for  $x_1$ ,  $x_2$  and  $x_3$ . This result is shown in Figure 4. The condition (11) in this case is equal to 3.95.

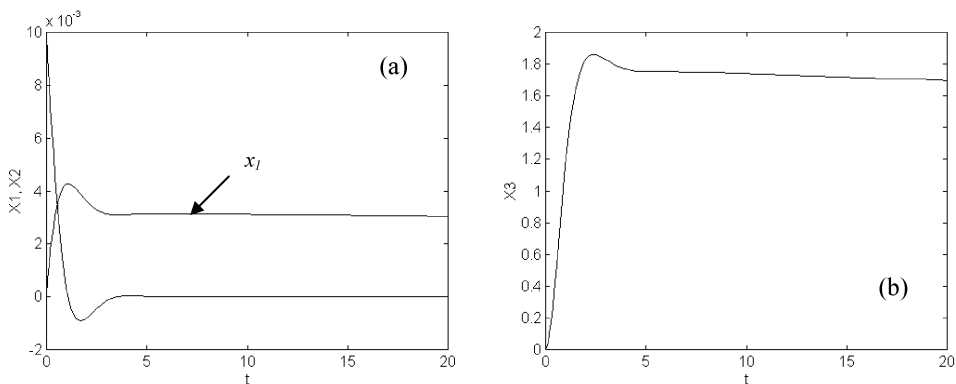


Fig.3: Stable solutions of the system (1) at  $V = 55.76$  m/s, i.e. 200 km/h

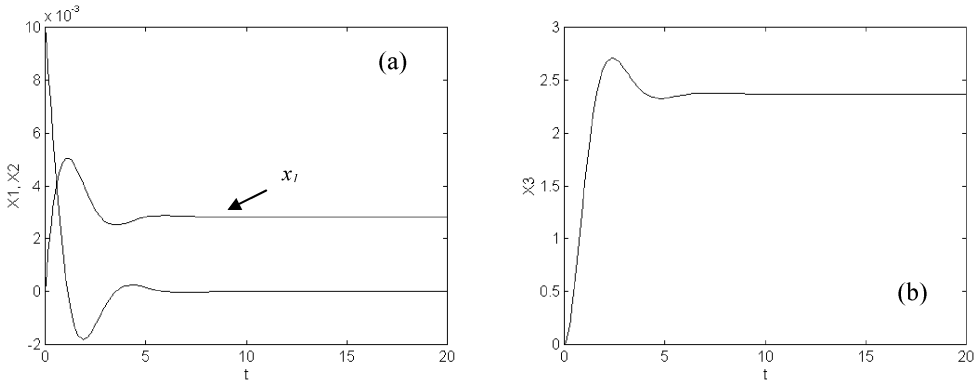


Fig.4: Stable solutions for  $x_1$ ,  $x_2$  (a) and  $x_3$  (b) at  $A_1 = -1.7585$ ,  $A_2 = 0.9533$ ,  $A_3 = -0.0021$ ,  $A_4 = 9.92 \times 10^{-4}$ ,  $A_5 = 0.6842$ ,  $A_6 = 572.667$ ,  $A_7 = -0.3027$ ,  $A_8 = -0.019$ . Here, the automobile velocity is  $V = 83.34$  m/s, i.e. 300 km/h

The numerical results in Figures 2–4 need additional comments. The behavior of the system (1) depends on a condition (11), the sign of which, characterizes its stability or instability. It is seen in Figure 2, 3 and 4 that the behavior of the system (1) is stable, but amplitudes of  $x_1$ ,  $x_2$  and  $x_3$  are increased when the automobile velocity is greater. In other words, when the velocity is greater, the system is closer to boundary of stability. For example: if  $V = 100$  km/h then  $R = 37.59$ ; if  $V = 200$  km/h then  $R = 7.94$ , if  $V = 300$  km/h then  $R = 3.95$ . Here, we note that all numerical simulations in Figures 2–4 were made when  $K_1 a_1 - K_2 a_2 < 0$ .

In Figure 5, we fix  $A_2 = 1.43$ ,  $A_3 = -0.00315$ ,  $A_4 = 9.92 \times 10^{-4}$ ,  $A_5 = 1.03$ ,  $A_6 = 572.667$ ,  $A_7 = -0.454$  and  $A_8 = -0.019$  and vary the parameter  $A_1$  (i.e.  $\alpha_1$ ). For each  $A_1$  we plot the solution of  $x_1$ . We see that for smaller value  $A_1 = -1.818$  (i.e., for a smaller coefficient  $\alpha_1$ ) the solution faster decrease with respect to these obtained at  $A_1 = -1.7535$  and  $A_1 = -1.7502$ . Here we note that for these values of parameters  $A_1$  the system (1) lies in the region of stability of its parametric space.

The next Figure 6 shows the change of  $x_1$  as parameter  $A_6$  (i.e.  $\chi_1$ ) changes. In this case, fixed parameters are  $A_1 = -1.7535$ ,  $A_2 = 1.43$ ,  $A_3 = -0.00315$ ,  $A_4 = 9.92 \times 10^{-4}$ ,  $A_5 = 1.03$ ,  $A_7 = -0.454$ , and  $A_8 = -0.019$  i.e. the automobile velocity is 200 km/h. Here, we note that the system (1) is also stable.

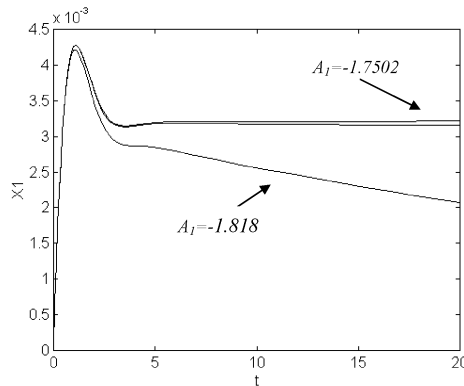


Fig.5: Three stable solutions for  $x_1$  at  $A_1 = -1.818$ ;  $-1.7535$  and  $-1.7502$

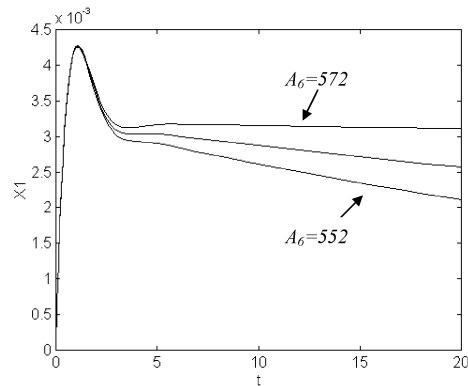


Fig.6: Three stable solutions for  $x_1$  at  $A_6 = 552$ ;  $562$  and  $572$

Comparing Figures 2–6, we conclude that in all cases the equilibrium state of the system (1) is stable. In the other words, when  $K_1 a_1 - K_2 a_2 < 0$ , the behavior of the automobile is always stable. This conclusion is in accordance with the Theorem proofed in [1].

Let us now describe the cases when  $K_1 a_1 - K_2 a_2 > 0$ . Here, we consider the examples for which the system (1) has only unstable solutions, i.e.  $R$  (in Eq. (11)) is negative. When the Routh-Hurwitz condition for stability (11) is negative the steady state of the system (1) becomes unstable. In order to define the type of stability loss ('soft' or 'hard') of the steady state ( $x_1^s = 0$ ,  $x_2^s = 0$ ,  $x_3^s = 0$ ) it is necessary to calculate the so-called first Lyapunov value [14, 15]. In previous paper [6], we obtain the analytical form of the first Lyapunov value  $L_1(\lambda_0)$  on the boundary of stability  $R = 0$  in equation (23) – see Eq. (A.3) in Appendix. Using (23) from [6], we plot of scale  $\alpha_1$ ,  $\alpha_2$  versus  $L_1$  or  $\chi_1$ ,  $\chi_2$  versus  $L_1$ .

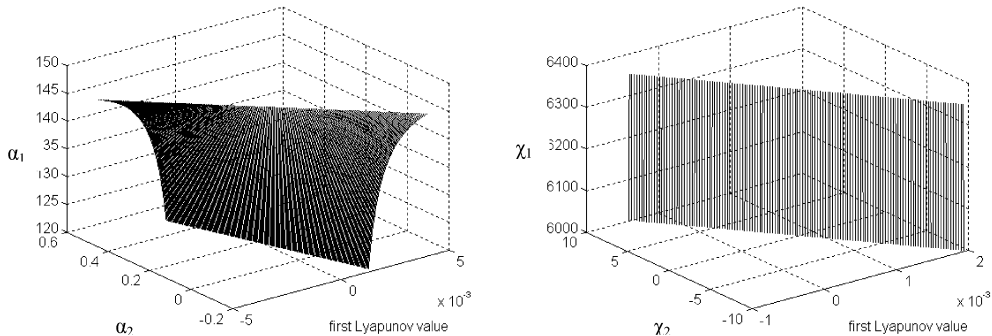


Fig.7: Dependence of the first Lyapunov value  $L_1$  on the parameters  $\alpha_1$ ,  $\alpha_2$  (left panel) and on the parameters  $\chi_1$ ,  $\chi_2$  (right panel); for other details, see text

In Figure 7 (left panel)  $L_1(\lambda_0)$  is shown for different values of the bifurcation parameters  $\alpha_1$  and  $\alpha_2$  when  $A_2 = 1.3667$ ,  $A_3 = 0.0022$ ,  $A_5 = 0.9809$ ,  $A_6 = 604.953$ ,  $A_7 = 3.71$  and  $A_8 = -0.0238$ . Here we note that these numerical values of  $A_2$ ,  $A_3$  and  $A_5$  to  $A_8$  are obtained at the automobile velocity  $V = 211$  km/h. It can be seen that  $L_1(\lambda_0)$  passes through regions for which it is negative or positive. Figure 7 (right panel) shows the dependence of the first Lyapunov value  $L_1(\lambda_0)$  on the parameters  $\chi_1$  and  $\chi_2$ . It is evident that  $L_1(\lambda_0)$  also passes through regions for which it is negative or positive, i.e. the car's stability loss can be reversible ('soft') or irreversible ('hard'). Here we note that in this case (automobile

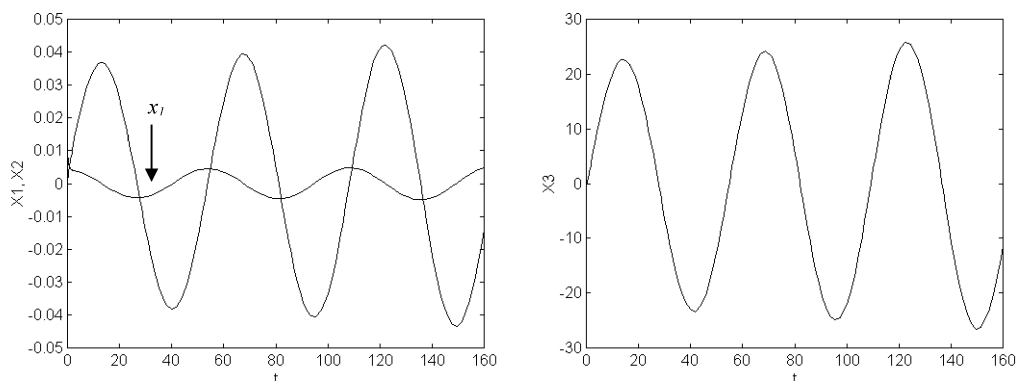


Fig.8: The angles of deviation  $x_1$  and the angular velocity  $x_2$  (left panel) and the cross velocity  $x_3$  (right panel) as function of time when the Routh-Hurwitz condition for stability  $R$  (Eq. (11)) is negative; here  $A_1 = 1.3249$ ,  $A_2 = 1.3667$ ,  $A_3 = 0.0022$ ,  $A_4 = 1.25 \times 10^{-4}$ ,  $A_5 = 0.9809$ ,  $A_6 = 604.95$ ,  $A_7 = 3.71$ ,  $A_8 = -0.0238$

stability) soft stability loss can be connected with the possibility of the driver to rid of car crash. For example, after decreasing of the velocity or a good reaction. On the other hand, in the case of hard stability loss the car crash is always valid.

Figure 8 depicts the case when  $R$  is negative ( $R = -0.0062$ ). The left panel demonstrates the angles of deviation and the angular velocity time behavior, and the right panel – the cross velocity time behavior. It is evident that in this case the system (1) has unstable solutions.

## (ii) Ford Focus

Following the same procedure, firstly we investigate the behavior of the system (1) when  $K_1 a_1 - K_2 a_2 < 0$ . After substitution of the second row values (see Table 1) into (2), for the dimensionless coefficients  $A_1$  to  $A_8$  we can write

$$\begin{aligned} A_1 &= -0.7214, & A_2 &= 2.24, & A_3 &= -0.0024, & A_4 &= 0.0045, \\ A_5 &= 2.143, & A_6 &= 598.667, & A_7 &= -0.3452, & A_8 &= -0.6667, \end{aligned} \quad (12)$$

where  $a_1 = a_2 = 1.3075$  m,  $K = K_1 = 22050$  N,  $K_2 = 22600$  N,  $V = 27.78$  m/s.

Figure 9 shows  $x_1$ ,  $x_2$  and  $x_3$  at  $V = 27.78$  m/s and Figure 10 shows  $x_1$ ,  $x_2$  and  $x_3$  at  $V = 55.56$  m/s. It is important to note here that for  $V = 27.78$  and  $V = 55.56$  m/s the system (1) has stable solutions i.e. the Routh-Hurwitz conditions for stability (8)–(11) are positive but in second case ( $V = 55.56$  m/s) the system (1) is closer to boundary of stability. For example, at  $V = 27.78$  m/s, the Routh-Hurwitz condition for stability  $R$  (Eq. (11)) is 24.1083 and at  $V = 55.56$  m/s  $R = 4.1559$ .

Figure 11 illustrates the dependence of the first Lyapunov value  $L_1(\lambda_0)$  on the parameters  $\alpha_1$  and  $\alpha_2$  (left panel) and on the parameters  $\chi_1$  and  $\chi_2$  (right panel). It should be remarked that the positive and negative regions take place, i.e. the sign of  $L_1(\lambda_0)$  changes. In the other words, if  $L_1(\lambda_0)$  is positive, then we have hard loss of stability and the system (1) has irreversible behavior, and if  $L_1(\lambda_0)$  is negative the reversible (soft loss of stability) behavior of the system (10) take place, see [6] for details. Here we note that the Routh-Hurwitz condition for stability  $R$  (Eq. (11)) is negative and automobile velocity  $V$  is 120 km/h and  $K_1 a_1 - K_2 a_2 > 0$ .

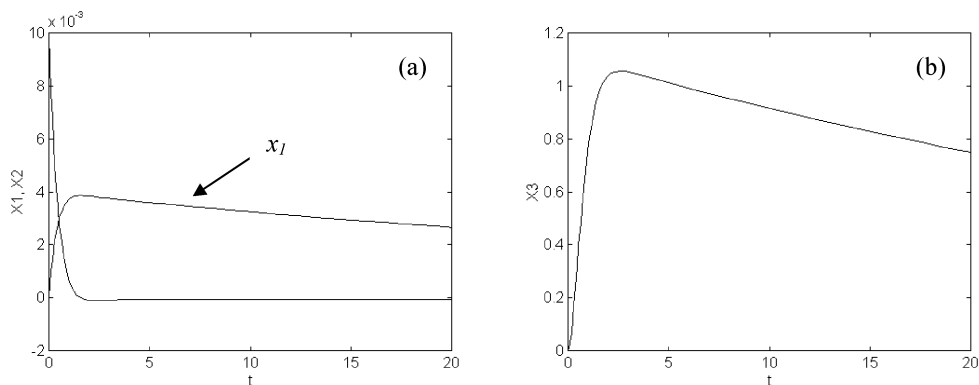


Fig.9: Stable solutions for  $x_1$ ,  $x_2$  (a) and  $x_3$  (b) at  $A_1 = -0.7214$ ,  $A_2 = 2.2413$ ,  $A_3 = -0.0024$ ,  $A_4 = 0.0045$ ,  $A_5 = 2.143$ ,  $A_6 = 598.667$ ,  $A_7 = -0.3452$ ,  $A_8 = -0.6667$ ; here, velocity is  $V = 27.78$  m/s, i.e. 100 km/h

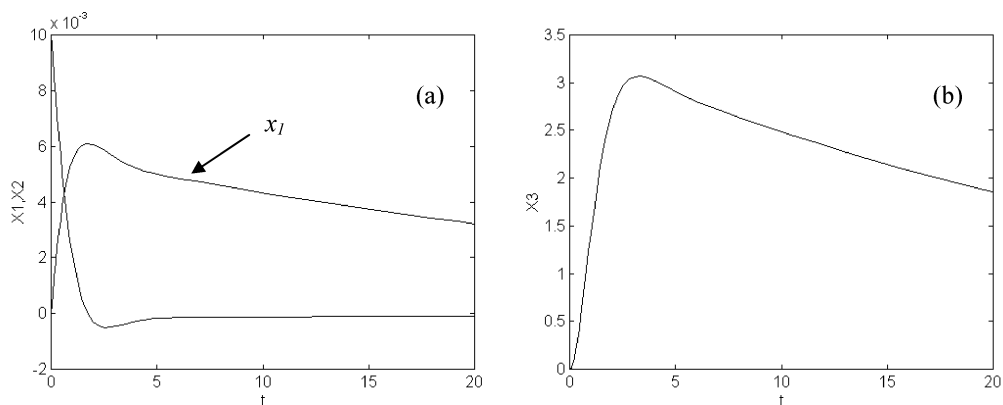


Fig.10: Stable solutions for  $x_1$ ,  $x_2$  (a) and  $x_3$  (b) at  $V = 55.56$  m/s, i.e. 200 km/h

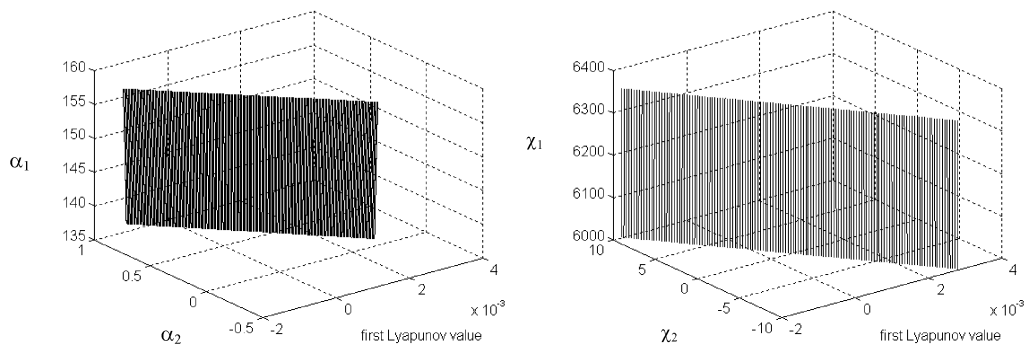


Fig.11: Dependence of  $L_1$  on the parameters  $\alpha_1$ ,  $\alpha_2$  (left panel) and on the parameters  $\chi_1$ ,  $\chi_2$  (right panel)

Figure 12 depicts the case when the system (1) has unstable solutions i.e.  $R$  is negative. The left panel demonstrates the angles of deviation and the angular velocity time behavior, and the right panel – the cross velocity time behavior.



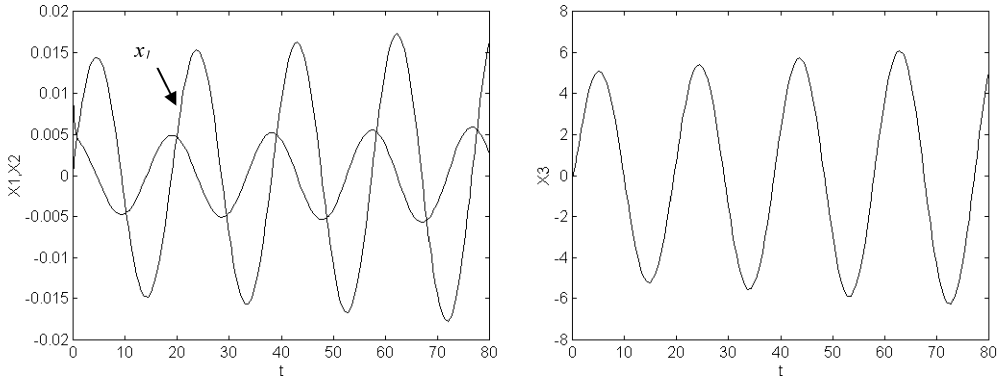


Fig.12: The angles of deviation  $x_1$  and the angular velocity  $x_2$  (left panel) and the cross velocity  $x_3$  (right panel) as function of time when the Routh-Hurwitz condition for stability  $R$  (Eq. (11)) is negative; here  $A_1 = 3.0825$ ,  $A_2 = 1.9566$ ,  $A_3 = 0.0092$ ,  $A_4 = -6.94 \times 10^{-5}$ ,  $A_5 = 1.6481$ ,  $A_6 = 594.067$ ,  $A_7 = 6.3792$ ,  $A_8 = 0.0663$

### (iii) Mercedes-Benz Sprinter (1)

Finally, we investigate the behavior of the system (1) when  $K_1 a_1 - K_2 a_2 < 0$ . After substitution of the third row values (see Table I) into (2), for the dimensionless coefficients  $A_1$  to  $A_8$  we can write

$$\begin{aligned} A_1 &= -0.3998, & A_2 &= 3.3258, & A_3 &= -0.0012, & A_4 &= 0.0053, \\ A_5 &= 2.1285, & A_6 &= 594.27, & A_7 &= -0.178, & A_8 &= -0.523, \end{aligned} \quad (13)$$

where  $a_1 = a_2 = 1.5$  m,  $K = K_1 = 54390$  N,  $K_2 = 55000$  N,  $V = 27.78$  m/s.

Figure 13 demonstrates the dependence of the solutions of the system (1) on the automobile velocity  $V$ , i.e. when  $V = 27.78$  m/s,  $V = 55.56$  m/s and  $V = 77.784$  m/s. We see that for larger value  $V = 77.784$  m/s the oscillation magnitudes are also larger to those obtained at  $V = 27.78$  m/s and  $V = 55.56$  m/s, i.e. for  $V = 77.784$  m/s the system (1) is closer to boundary of stability. Here we note that in these cases the Routh-Hurwitz conditions for stability are always positive.

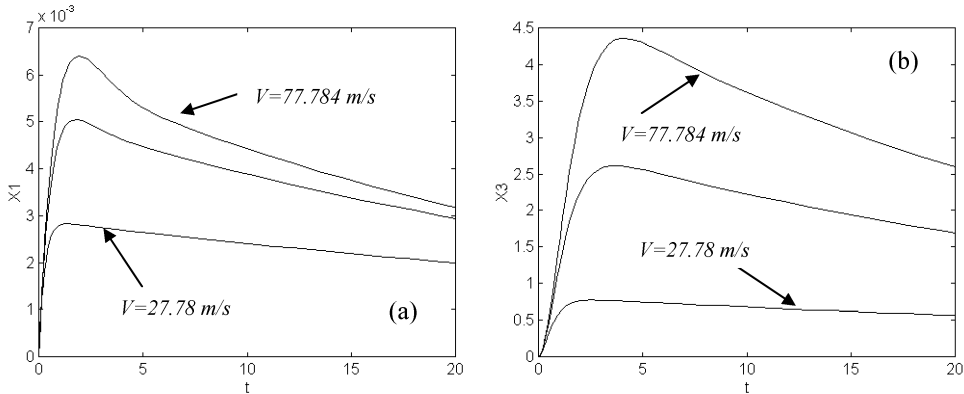


Fig.13: Stable solutions for  $x_1$  (a) and  $x_3$  (b) at  $V = 27.28$  m/s,  $V = 55.56$  m/s and  $V = 77.784$  m/s

In Figure 14 (left panel) we show  $L_1(\lambda_0)$  for different values of the bifurcation parameters  $\alpha_1$  and  $\alpha_2$ . It can be seen that  $L_1(\lambda_0)$  passes through regions for which it is negative or positive. Figure 14 (right panel) shows the dependence of the first Lyapunov value  $L_1(\lambda_0)$  on the parameters  $\chi_1$  and  $\chi_2$ . It is evident that  $L_1(\lambda_0)$  also passes through regions for which it is negative or positive. Here, the automobile velocity is  $V = 225.5 \text{ km/h}$  and  $K_1 a_1 - K_2 a_2 > 0$ .

In Figure 15 the case when the system (1) has unstable solutions i.e.  $R$  negative is shown. The left panel demonstrates the angles of deviation and the angular velocity time behavior, and the right panel- the cross velocity time behavior.

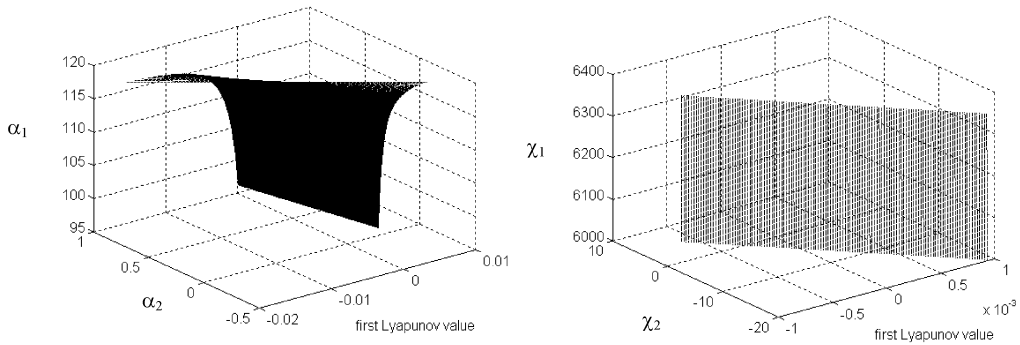


Fig.14: Dependence of  $L_1$  on the parameters  $\alpha_1, \alpha_2$  (left panel) and on the parameters  $\chi_1, \chi_2$  (right panel)

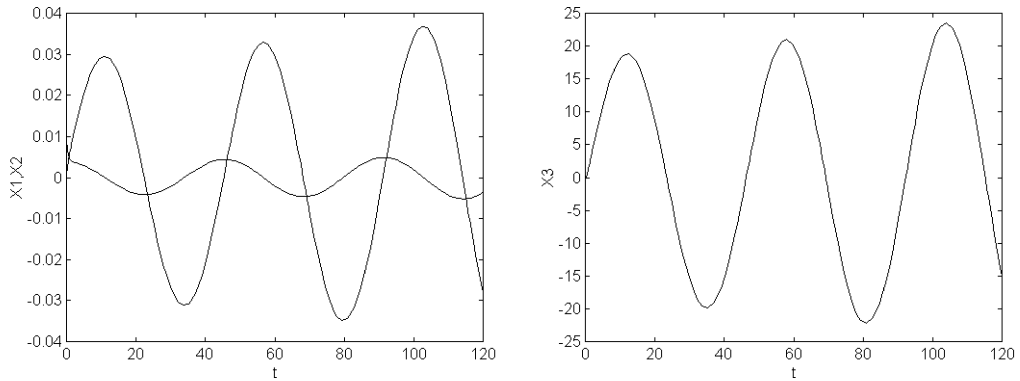


Fig.15: The angles of deviation  $x_1$  and the angular velocity  $x_2$  (left panel) and the cross velocity  $x_3$  (right panel) as function of time when the Routh-Hurwitz condition for stability  $R$  (Eq. (11)) is negative; here  $A_1 = 1.3044$ ,  $A_2 = 1.4357$ ,  $A_3 = 0.0021$ ,  $A_4 = 1.8769 \times 10^{-5}$ ,  $A_5 = 0.9189$ ,  $A_6 = 591.8378$ ,  $A_7 = 3.6681$ ,  $A_8 = 0.0135$

### 3. Discussion and conclusions

In this paper we consider an idealized mathematical model of the roadway automobile motion. This model is intended to describe the roadway automobile stability and was developed by us in [6]. The specific question we attempted to answer is: which parameter(s) of automobile geometry and quality of the roadway can serve as a reliable predictor(s) for car

crash? The model equations, represented by Eq. (1), are solved numerically and analyzed by theory of Lyapunov-Andronov.

The dynamic model (1), formulated in our previous paper [6], as well as related analytical conditions of stability and first Lyapunov's value  $L_1$ , were checked numerically in Section 2 of the present work. Data from [8–10] were used for that purpose, considering three car types – Mercedes-Benz E 320 (T-modelle), Ford Focus and Mercedes-Benz Sprinter (1). Hence, one can consider the present work as a natural continuation of [6]. It is seen in Figs. 2–6, Fig. 9, Fig. 10 and Fig. 13 that when  $K_1 a_1 - K_2 a_2 < 0$ , motion is always stable (which is in accordance with theorem for stability of automobile proofed in [1]), regardless of car speed and values of the coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$ . For instance, considering Mercedes-Benz E 320 (T-modelle), the car is 'less stable' under speed of 300 km/h, as compared to its stability under speed of 100 km/h. In terms of our model, this is linked with the stability condition  $R$  (Eq. (11)), and it is that  $R = 37.59$  under 100 km/h and  $R = 3.95$  under 300 km/h, i.e. we are closer to the stability limit. These results are in accordance with the theorem proved by Rokard in [1]. Note that the numerical simulations of the effect of coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$  in this case ( $K_1 a_1 - K_2 a_2 < 0$ ) are performed, using data for Mercedes-Benz E 320 (T-modelle), only. This is so, since one can find similar results for the other types of cars, i.e. we have stable solutions, only.

Considering a dynamic point of view, the case when  $K_1 a_1 - K_2 a_2 > 0$  is more interesting. (This can be attained for: (i)  $K_1 > K_2$ ,  $a_1 = a_2$ ; (ii)  $K_1 < K_2$ ,  $a_1 \gg a_2$ ; (iii)  $K_1 = K_2$ ,  $a_1 > a_2$ ; (iv)  $K_1 > K_2$ ,  $a_1 > a_2$ . Our studies concern case (i)). Under such condition, car motion can be either stable or unstable. On the other hand, unstable motion can be reversible – soft stability loss, and irreversible – hard stability loss. The character of stability loss depends on the sign of the first Lyapunov's value [14–17].

The results found and shown in Fig. 7, Fig. 11 and Fig. 14, illustrate how the sign of  $L_1$  changes under fixed speed and varying  $\alpha_1$ ,  $\alpha_2$  or  $\chi_1$ ,  $\chi_2$ , when  $R < 0$  (i.e. when motion is unstable). Considering Mercedes-Benz E 320 (T-modelle), the speed of unstable motion and  $L_1 > 0$ , is 211 km/h. Considering Ford-Focus, it is 120 km/h, while for Mercedes-Benz Sprinter (1) it is 225.5 km/h (Note that the speed of 225.5 km/h can never be attained in practice). The important point here is that coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$  are approximately one and the same for all three types of cars, i.e. car behaviour depends on car characteristics, only. Our conclusion is that cars with the greatest mass and base attain hard stability loss (car crash) under higher speeds. This is Mercedes-Benz Sprinter (1) in our case, where  $V = 225.5$  km/h. On the contrary, cars with small mass and base attain more easily (with lower speed) irreversible stability loss. Such car is Ford Focus here, whose stability loss is irreversible for a speed of 120 km/h.

Another important result is that, except for speed, car stability depends essentially on the coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$ . For instance, for other values of the coefficients (greater or smaller than the given ones), the speed for which  $L_1 > 0$  would be different from the one, found here for all three cases. This result is a confirmation of Conclusions 1 and 3 of [5], namely, that the qualitative behaviour of model (1) depends essentially on the coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\chi_1$  and  $\chi_2$  which define the sign of  $L_1$ . For instance, Fig. 9, Fig. 11 and Fig. 15 show solutions of system (1) under hard stability loss.

In conclusion, we note the following: our results are found on the basis of the investigation of a basic (qualitative) dynamic model – a model of 1 or 2 degrees of freedom. Hence, our

basic task is not to compare separate types of cars, but to outline qualitative tendencies, which car full stability would follow. This means that we assess the essential effect of the discussed parameters on car qualitative behaviour.

## References

- [1] Rocard Y.: Mechanical instability: automobiles, airplanes, suspension bridges, Paris, Masson et Cie, Editeurs, 1954 (in French)
- [2] Chudakov E.A.: Theory of automobile, Nauka, Moscow, 1961 (in Russian)
- [3] Litvinov A.S., Farobin Ia.E.: Automobile. Theory of the service attribute, Maschinostroene, Moscow, 1989 (in Russian)
- [4] Abadjiev V., Gospodinov P.: Identification of the velocity characteristics of a car crash, Journal of Theoretical and Applied Mechanics, vol. 29, No 4, pp.3–12, 1999
- [5] Abadjiev V., Gospodinov P., Abadjieva E.: Car crash passenger place identification, Archives of Transport, vol. 13, No 1, pp. 5–14, 2001
- [6] Nikolov S., Bachvarov S.: An analytical study of the roadway automobile stability, System Analysis-Modelling-Simulation (SAMS), vol. 42, No 8, pp.1271–1281, 2002
- [7] Orosz G., Wilson R., Krauskopf B.: Global bifurcation investigation of an optimal velocity traffic model with driver reaction time, Phys. Rev. E, vol. 70, pp. 026207-11, 2004
- [8] Daimler-Chrysler AG, Stuttgart MKP/K 6701-0507-00-04/0701 Printed in Federal Republic of Germany, pp. 56–57
- [9] Daimler-Chrysler AG, Stuttgart VT/KG 633-11301-01-01/0301 Printed in Federal Republic of Germany, 24
- [10] Ford Motor Company Limited (March 2001), PN 172501/0103/7.225m/Engex LHD, Printed by Mairs Graphische Betriebe ·D-73760 Ostfildern, Federal Republic of Germany, Published by Ford Automotive Operations – Europe, Marketing Communications, 30
- [11] Mitschke M.: Dynamik der Kraftfahrzeuge, Band A: Antrieb und Bremsung, Springer-Verlag, Berlin, 1995
- [12] Mitschke M.: Dynamik der Kraftfahrzeuge, Band B: Schwingungen, Springer-Verlag, Berlin, 1997
- [13] Mitschke M.: Dynamik der Kraftfahrzeuge, Band C: Fahrverhalten, Springer-Verlag, Berlin, 1990
- [14] Bautin N.N.: Behavior of dynamical systems near the boundary of stability. Nauka, Moscow, 1984 (in Russian)
- [15] Nikolov S.: First Lyapunov value and bifurcation behavior of specific class of three-dimensional systems, Int. J. of Bifurcation and Chaos, vol. 14, No 8, pp. 293–308, 2004
- [16] Nikolov S., Petrov V.: New results about route to chaos in Rossler system, Int. J. of Bifurcation and Chaos, vol. 14, No 1, pp. 2811–2823, 2004
- [17] Andronov A., Witt A., Chaikin S.: Theory of Oscillations, Addison-Wesley, Reading, MA, 1966
- [18] Shilnikov L., Shilnikov A., Turaev D., Chua L.: Methods of Qualitative Theory in Nonlinear Dynmics, Part II, World Scientific, Singapore, 2001
- [19] Panovko Y., Gubanova I.: Stability and Oscillations of Elastic Systems, Consultant bureau, NY, 1965
- [20] Litvinov A.: Automobile Stability and Control, Maschinostroene, Moscow, 1971 (in Russian)

## Appendix

### I. Model

From the fundamental (natural) point of view, the external side force  $F$  and the external moment  $C$  which changing the direction of the automobile wheels are different from zero. This hypothesis follows from experimental works [2, 3]. Because of that we propose the external side force  $F(\theta)$  and external moment  $C(\theta)$  to be polynomial functions, which we

can determine them by using Taylor series. According to [2, 3, 19, 20], we assume for  $F(\theta)$  and  $C(\theta)$  the form

$$\begin{aligned} F(\theta) &= \chi_1 \theta + \chi_2 \theta^3, \\ C(\theta) &= \alpha_1 \theta + \alpha_2 \theta^3, \end{aligned} \quad (\text{A.1})$$

where  $\chi_1, \chi_2$  are constants having the dimension of force and  $\alpha_1, \alpha_2$  are constants with moment dimension.

## II. Calculation of the first Lyapunov value

In the our previous paper [6], following [14], we calculate the first Lyapunov value (this is not Lyapunov exponent-see appendix in [16] or for a detailed discussion (Andronov et al. 1966, Shilnikov et al. 2001, Nikolov 2004) at the boundary of stability region  $R = 0$  of the system (1). Generally, in accordance with Lyapunov-Andronov theory we have: (i) the sign of Lyapunov's value determines the character (stable or unstable) of equilibrium state at  $R = 0$ ; (ii) the character of equilibrium state, at  $R = 0$  qualitatively determines the reconstruction of phase space (including stability or instability of limit cycle) at the transition from  $R < 0$  to  $R > 0$ .

In the case of three first-order nonlinear differential equations, this value can be determined analytically by the formula in [14]

$$\begin{aligned} L_1(\lambda_0) &= \frac{\pi}{4q} \left[ 2 \left( A_{33}^{(2)} A_{33}^{(3)} - A_{22}^{(2)} A_{22}^{(3)} \right) + 2 A_{23}^{(2)} \left( A_{22}^{(2)} + A_{33}^{(2)} \right) - \right. \\ &\quad \left. - 2 A_{23}^{(3)} \left( A_{22}^{(3)} + A_{33}^{(3)} \right) + 3 \sqrt{q} \left( A_{222}^{(2)} + A_{333}^{(3)} + A_{233}^{(2)} + A_{223}^{(3)} \right) \right] + \\ &\quad + \frac{\pi}{4p\sqrt{q}(p^2 + 4q)} \left\{ p^2 \left[ 2 A_{22}^{(1)} \left( A_{12}^{(2)} + A_{13}^{(3)} \right) + \right. \right. \\ &\quad \left. \left. + 2 A_{33}^{(1)} \left( A_{12}^{(2)} + 3 A_{13}^{(3)} \right) + 4 A_{23}^{(1)} \left( A_{13}^{(2)} + A_{12}^{(3)} \right) \right] + \right. \\ &\quad \left. + 4p\sqrt{q} \left[ \left( A_{22}^{(1)} - A_{33}^{(1)} \right) \left( A_{13}^{(2)} + A_{12}^{(3)} \right) + 2 A_{23}^{(1)} \left( A_{13}^{(3)} - A_{12}^{(2)} \right) \right] + \right. \\ &\quad \left. + 16q \left( A_{22}^{(1)} + A_{33}^{(1)} \right) \left( A_{12}^{(2)} + A_{13}^{(3)} \right) \right\}, \end{aligned} \quad (\text{A.2})$$

where  $\lambda_0$  is defined as a value of  $\alpha_1, \alpha_2, \chi_1, \chi_2, K_1, K_2, a_1, a_2$  and car velocity  $V$  for which the relation  $R = 0$  takes place. The coefficients  $A_{ij}^n$  and  $A_{ijk}^n$  ( $i, j, k, n = 1, 2, 3$ ) are defined by corresponding formulas presented in [14]. For the system (1)  $A_{ij}^n = 0$ . Thus, after accomplishing some transformations and algebraic operations for the first Lyapunov value  $L_1(\lambda_0)$  we obtain:

$$\begin{aligned} L_1(\lambda_0) &= -\frac{3\pi}{4\sqrt{q}} (\alpha_{12}^2 + \alpha_{13}^2) \times \\ &\quad \times \frac{A_1(A_1 A_8 - A_4 A_6) + p[A_4(A_2 A_6 - A_1 A_7) + pq A_8]}{(p^2 + A_1)(A_2 A_6 - A_1 A_7) + \frac{1}{A_3}(p A_5 - A_1) + p A_6(q - A_1)}, \end{aligned} \quad (\text{A.3})$$

where the coefficients  $\alpha_{12}$  and  $\alpha_{13}$  are defined also in [14] and for the system they are

$$\alpha_{12} = -A_2, \quad \alpha_{13} = -p\sqrt{q} = -(A_2 + A_5)\sqrt{q}. \quad (\text{A.4})$$

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