CRACK INITIATION CRITERIA FOR SINGULAR STRESS CONCENTRATIONS
Part III: An Application to a Crack Touching a Bimaterial Interface

Luboš Náhlík*, Zdeněk Knésl, Jan Klusák**

The paper deals with crack propagation through an interface between two elastic materials. The basic idea of developing stability criteria of general singular stress concentrators introduced in the first part is applied to the case of a crack with its tip at the interface between two different materials. Three different stability criteria based on different physical principles are presented and a numerical example with their mutual comparison is carried out. A procedure based on a generalized strain energy density factor is shown which makes it possible to estimate the further direction of crack propagation after the crack has passed the interface. The procedure presented is applied in the numerical examples.

Key words : *bimaterial interface, stability criteria, crack propagation, critical stress, threshold stress*

1. Introduction

The advanced materials, such as fiber or particle reinforced composites, metal-ceramic interfaces, laminated ceramics, adhesive joints, protective layers etc. have many important applications in the industry. Their behavior is strongly influenced by the existence of material interfaces. Interface failures are common features in these materials and the design process requires a better understanding of the corresponding failure mechanisms. It is an important task to make detailed investigations into the fracture characteristics of cracks propagating through bimaterial interfaces.

An interface between two dissimilar media represents a weak point for many applications of structures composed of different materials. The presence of regions with different mechanical properties and the existence of an interface between them have a pronounced influence on the stress distribution of composite bodies. The characteristics of fracture in the vicinity of and at the interface are strongly influenced by the properties of the interface and of the material on either side of the interface. A basic problem is to assess the influence of the interface on a crack penetrating from one material into the other in a bimaterial body.

The concentration of stress due to interface defects has recently been studied extensively and basic ideas of the fracture mechanics of interfaces have been developed, but many

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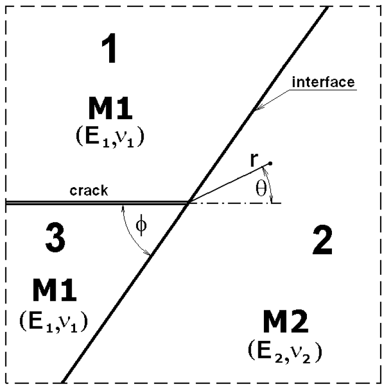

Fig.1: A crack touching the interface between two materials (M1 and M2)

questions are still to be answered [1–6]. The propagation of cracks which approach and intersect the interfaces between dissimilar solids is a topic of considerable academic and practical interest. In the past most of the investigations dealt with crack growth near interfaces [7–10].

In this paper we report the results of a theoretical study concerning mainly the behavior of a crack with its tip at the interface. This contribution directly relates to the preceding papers [11, 12] (in the following referred to as Part I and Part II, respectively). The configuration studied here is shown in Fig. 1. The problem is studied under the assumptions of linear elastic fracture mechanics (LEFM) and the interface is assumed of a welded type (ideal adhesion is considered). The fact that in the studied cases the stress singularity exponent(s) are different from 1/2 ($p \neq 1/2$) complicates the formulation of stability criteria. To solve the problem, the general approach suggested and formulated in Part I of the contribution is applied.

2. Stress distribution around the crack tip

The stress distribution around the crack tip can be determined by means of a combination of analytical and numerical approaches [13–15]. The common detailed procedure based on the Airy stress function is introduced in Part II (see equations 1–10). The geometric configuration studied here differs from that considered in Part II mainly in the formulation of the boundary conditions. The solutions for the biharmonic stress function for the configuration shown in Fig. 1 are sought such that the crack faces ($\theta = \pm\pi$) are traction-free for all values of r [14]. In terms of the stress components the condition requires that (the first index relates to number of the region, see Fig. 1)

$$\sigma_{1\theta\theta}(r,\pi) = \sigma_{1\mathrm{r}\theta}(r,\pi) = \sigma_{3\theta\theta}(r,\,-\pi) = \sigma_{3\mathrm{r}\theta}(r,\,-\pi) = 0\ . \tag{1}$$

Perfect bonding along the interface $\theta = \phi$, $\theta = \phi - \pi$ is ensured by the following stress and displacement continuity conditions

$$u_{1\mathrm{r}}(r,\phi) = u_{2\mathrm{r}}(r,\phi)\ , \qquad u_{3\mathrm{r}}(r,\phi-\pi) = u_{2\mathrm{r}}(r,\phi-\pi)\ ,$$
$$u_{1\theta}(r,\phi) = u_{2\theta}(r,\phi)\ , \qquad u_{3\theta}(r,\phi-\pi) = u_{2\theta}(r,\phi-\pi)\ ,$$

and

$$\begin{aligned}\sigma_{1\theta\theta}(r,\phi) &= \sigma_{2\theta\theta}(r,\phi)\ , \qquad \sigma_{3\theta\theta}(r,\phi-\pi) = \sigma_{2\theta\theta}(r,\phi-\pi)\ ,\\ \sigma_{1\mathrm{r}\theta}(r,\phi) &= \sigma_{2\mathrm{r}\theta}(r,\phi)\ , \qquad \sigma_{3\mathrm{r}\theta}(r,\phi-\pi) = \sigma_{2\mathrm{r}\theta}(r,\phi-\pi)\ .\end{aligned} \tag{2}$$

These twelve conditions lead to the system of 12 linear equations $\mathbf{B}(\lambda)\,\mathbf{x} = \mathbf{0}$, where $\mathbf{B}(\lambda)$ is the matrix of the system and $\mathbf{x}$ is the vector of 12 unknown coefficients. The system contains 13 unknowns in total. The thirteenth unknown is the eigenvalue λ which can be obtained from the condition of the existence of a nontrivial solution of the system, i.e.

$$\det \mathbf{B}(\lambda) = 0 \ . \tag{3}$$

Following the basic idea of Chapter 2, Part I, the general expression describing the stress field around the crack tip touching the interface between two materials (see Fig. 1) is given by the following equations (generally, two stress singularity exponents exist) :

$$\sigma_{ij} = \frac{H_1}{\sqrt{2\pi}}\, r^{-p_1}\, f_{1ij}(\theta, \alpha_\mathrm{D}, \beta_\mathrm{D}, \phi) + \frac{H_2}{\sqrt{2\pi}}\, r^{-p_2}\, f_{2ij}(\theta, \alpha_\mathrm{D}, \beta_\mathrm{D}, \phi) \ , \tag{4}$$

where H_1 [MPa m^{p_1}] and H_2 [MPa m^{p_2}] are generalized stress intensity factors, f_{1ij} and f_{2ij} are known functions, (r, θ) are polar coordinates with their beginning at the crack tip and p_1 and p_2 are stress singularity exponents. The values of p_1 and p_2 are determined from the condition (3) and $p_1 = 1 - \lambda_1$ and $p_2 = 1 - \lambda_2$ respectively. α_D, β_D are Dundurs parameters (see [16] for details) and represent elastic constants of a bimaterial body.

$$\alpha_\mathrm{D} = \frac{-\mu_1\,(\kappa_2+1) + \mu_2\,(\kappa_1+1)}{\mu_1\,(\kappa_2+1) + \mu_2\,(\kappa_1+1)} \ , \qquad \beta_\mathrm{D} = \frac{-\mu_1\,(\kappa_2-1) + \mu_2\,(\kappa_1-1)}{\mu_1\,(\kappa_2+1) + \mu_2\,(\kappa_1+1)} \ , \tag{5}$$

where the shear modulus $\mu_\mathrm{m} = E_\mathrm{m}/[2\,(1+\nu_\mathrm{m})]$ (E_m is Young's modulus) and the parameters $\kappa_\mathrm{m} = (3-\nu_\mathrm{m})/(1+\nu_\mathrm{m})$ for the case of plane stress or $\kappa_\mathrm{m} = 1 - 4\,\nu_\mathrm{m}$ for plane strain ($m = 1, 2$ corresponds to the number of the material).

Note that the values H_1 and H_2 do not mark the appurtenance to the normal or the shear mode of loading, but the values include both modes of crack propagation.

Generally, two singularity exponents p_1, p_2 lying within the interval $(0; 1)$ exist for the case of a crack touching the interface. For a crack perpendicular to the interface ($\phi = \pi/2$) only one real singularity exponent p exists in this interval. For the angle ϕ smaller than 90 degrees two real exponents exist (see Fig. 2). For small angles ϕ we can find only complex stress singularity exponents, see e.g. [13], [14] for details. For a crack touching the interface the general form of the stress components is given by the expressions (6).

$$\begin{aligned}
\sigma_\mathrm{rr} = \sum_{i=1}^{2} \frac{H_i}{\sqrt{2\pi}}\, r^{-p_i}\, (p_i-1)\, [a_i^{(j)}\,(2-p_i)\, \sin(2-p_i)\theta + b_i^{(j)}\,(2-p_i)\, \cos(2-p_i)\theta\, + \\
+\, c_i^{(j)}\,(-p_i-2)\, \sin(-p_i)\theta + d_i^{(j)}\,(-p_i-2)\, \cos(-p_i)\theta] \ , \\
\sigma_{\theta\theta} = \sum_{i=1}^{2} \frac{H_i}{\sqrt{2\pi}}\, r^{-p_i}\, (1-p_i)\,(2-p_i)\, r^{-p_i}\, [a_i^{(j)}\, \sin(2-p_i)\theta + b_i^{(j)}\, \cos(2-p_i)\theta\, + \\
+\, c_i^{(j)}\, \sin(-p_i)\theta + d_i^{(j)}\, \cos(-p_i)\theta] \ , \\
\sigma_{\mathrm{r}\theta} = \sum_{i=1}^{2} \frac{H_i}{\sqrt{2\pi}}\, r^{-p_i}\, (p_i-1)\, [a_i^{(j)}\,(2-p_i)\, \cos(2-p_i)\theta - b_i^{(j)}\,(2-p_i)\, \sin(2-p_i)\theta\, + \\
+\, c_i^{(j)}\,(-p_i)\, \cos(-p_i)\theta - d_i^{(j)}\,(-p_i)\, \sin(-p_i)\theta] \ .
\end{aligned} \tag{6}$$

The subscript $i = 1, 2$ means the number of singularities within the interval $(0;1)$. The superscript $j = 1, 2, 3$ marks the number of regions in Fig. 1. The constants a_j, b_j, c_j, d_j represent the known eigenvector $\mathbf{x}$, see e.g. [17] for details. The existence of more singularities complicates the formulation of stability criteria. However, in many practical cases a crack is perpendicular to the interface and there is only one real stress singularity.

Note that for an interfacial crack (i.e. for $\phi = 0$) the stress singularity exponent p is complex and has the form $p = 1/2 + \mathrm{i}\,\varepsilon$, where ε is a complex part of the solution of the equation (3), see [18, 19].

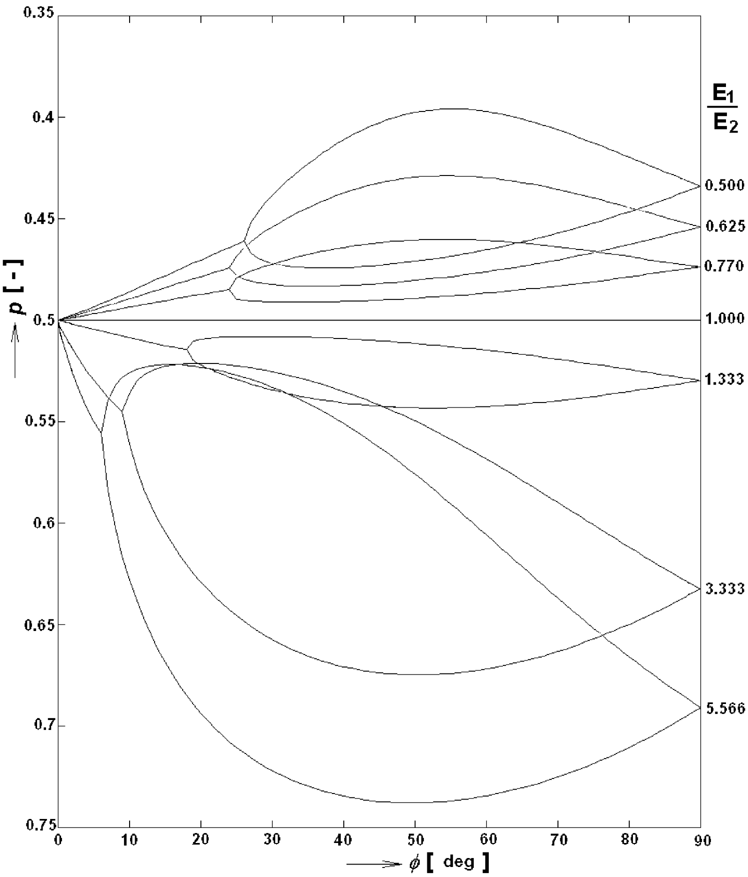

Fig.2: Real parts of the stress singularity exponents p in dependence on the orientation angle ϕ between the crack and the material interface for the selected ratio of Young's moduli [17]; Poisson's ratios of the materials were chosen $\nu_1 = \nu_2 = 0.3$

2.1. A crack perpendicular to the interface

The stress field around the crack tip (only one singularity exists) can be described by the expression:

$$\sigma_{ij} = H_{\mathrm{I}}\, r^{-p}\, f_{ij}(\theta, \alpha, \beta) \;, \tag{7}$$

where $f_{ij}(\theta, \alpha, \beta)$ is a known function and p is the corresponding stress singularity exponent. Its dependence on the ratio of Young's modulus ratio is shown in Fig. 3. The problems of

p and $H_{\rm I}$ determination were in detail described in Parts I and II of this work. Let us consider in the following that the crack propagates in mode I and under the conditions of plane strain.

In this case the equation (7) can be rewritten in the form (see e.g. [20]) :

$$\begin{aligned}
\sigma_{\rm xx} &= \frac{H_{\rm I}}{\sqrt{2\pi}}\, r^{-p}\,(1-p)\,[(2\,f_{\rm R}-g_{\rm R})\,\cos(-p)\theta-(2\,f_{\rm I}-g_{\rm I})\,\sin(-p)\theta+\\
&\qquad +p\,(f_{\rm R}\,\cos(-p-2)\theta-f_{\rm I}\,\sin(-p-2)\theta)]\ ,\\
\sigma_{\rm yy} &= \frac{H_{\rm I}}{\sqrt{2\pi}}\, r^{-p}\,(1-p)\,[(2\,f_{\rm R}+g_{\rm R})\,\cos(-p)\theta-(2\,f_{\rm I}+g_{\rm I})\,\sin(-p)\theta-\\
&\qquad -p\,(f_{\rm R}\,\cos(-p-2)\theta-f_{\rm I}\,\sin(-p-2)\theta)]\ ,\\
\sigma_{\rm xy} &= \frac{H_{\rm I}}{\sqrt{2\pi}}\, r^{-p}\,(1-p)\,[g_{\rm R}\,\cos(-p)\theta+g_{\rm I}\,\sin(-p)\theta-\\
&\qquad -p\,(f_{\rm R}\,\sin(-p-2)\theta+f_{\rm I}\,\cos(-p-2)\theta)]\ ,
\end{aligned}\tag{8}$$

where

$$f_{\rm R}=1\ ,\qquad f_{\rm I}=g_{\rm I}=0\ ,$$
$$g_{\rm R}=(1-p)-\cos(1-p)\pi-\beta\,\frac{\alpha+2\,(1-p)-[1+2\,\alpha-4\,\alpha\,(1-p)^2]\,\cos(1-p)\pi}{D(p)}\tag{9}$$

and

$$D(p)=1+2\,\alpha+2\,\alpha^2-2\,\alpha\,(1+\alpha)\,\cos(1-p)\pi-4\,\alpha^2\,(1-p)^2\ .$$

The constants α and β were introduced in [20] and characterize the elastic mismatch between the materials of the bimaterial. The bimaterial composite parameters α and β are conveniently defined in plane strain terms as

$$\alpha=\frac{\dfrac{\mu_1}{\mu_2}-1}{\eta_1+1}\qquad\text{and}\qquad\beta=\frac{\mu_1\,(\eta_2+1)}{\mu_2\,(\eta_1+1)}\ ,\tag{10}$$

where μ_1 and μ_2 are the shear moduli of the materials M1 and M2 respectively, ν_1, ν_2 are Poisson's ratio of both materials and $\eta_1=\eta_2=3-4\,\nu$ for plane strain conditions. Note that the definition of the parameters α and β used here differs from those used in Part II.

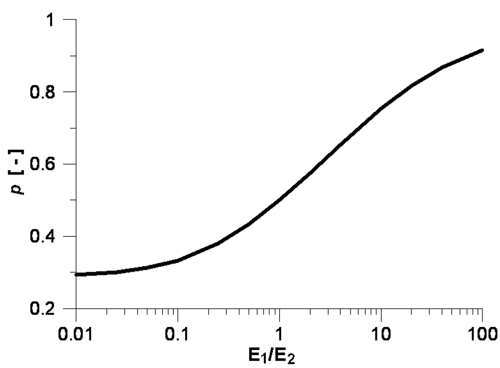

Fig.3: The stress singularity exponent for a crack perpendicular to the interface in dependence on the ratio of Young's moduli; Poisson's ratios are the same for both materials; ($\nu_1=\nu_2=0.3$)

3. Stability criteria

In the following three criteria based on the procedure intimately described in Part I are presented. As it is stated in Part I, the fact that for a crack with its tip at the interface between two different materials the stress singularity is no longer $r^{-1/2}$ means that the classical linear elastic fracture mechanics arguments and criteria cannot be applied.

3.1. Criterion based on average stress ahead of the crack tip

This stability criterion is suitable especially in the case of brittle fracture failure and for cracks perpendicular to the interface, where only one real singularity exists. The stability condition for a crack with an exponent of the singularity different from 1/2 is related to the average stress $\bar{\sigma}$ calculated across a distance d ahead of the crack tip [21], i.e. the quantity L (see Part I) is thus chosen as $L = \bar{\sigma}$, see Fig. 4.

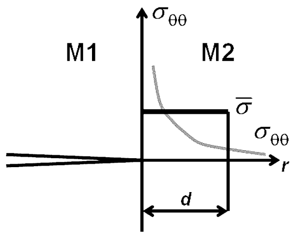

Fig.4: The average value of the stress component $\sigma_{\theta\theta}$ within the distance d in front of the crack tip

The propagation of the crack is controlled by the opening stress in the direction of the crack propagation, i.e. for $\theta = 0$. In this case the expression for the stress normal to the crack plane (the crack is considered parallel to the axis x) can be written in the following form (see eq. (8)):

$$\sigma_{\theta\theta}(r, \theta = 0) = \sigma_{\mathrm{yy}}(x, y = 0) = \frac{H_{\mathrm{I}}}{\sqrt{2\pi}} (1 - p)(2 - p + g_{\mathrm{R}})\, r^{-p} \, . \tag{11}$$

The average stress ahead of the crack tip is then given by the expression:

$$\bar{\sigma} = \frac{1}{d} \int_0^d \sigma_{\theta\theta}(r, \theta = 0)\, \mathrm{d}r \, . \tag{12}$$

For a crack in a homogeneous material it holds $\sigma_{\theta\theta}(r, \theta = 0) = K_{\mathrm{I}}/\sqrt{2\pi\, r}$ and

$$\bar{\sigma} = \frac{1}{d} \int_0^d \frac{K_{\mathrm{I}}}{\sqrt{2\pi\, r}}\, \mathrm{d}r = \frac{2\, K_{\mathrm{I}}}{\sqrt{2\pi\, d}} \, . \tag{13}$$

This value is related to the critical fracture stress $\bar{\sigma}_{\mathrm{crit}}$ which is a material constant and can be expressed by means of the fracture toughness K_{IC}:

$$\bar{\sigma}_{\mathrm{crit}} = \frac{1}{d} \int_0^d \sigma_{\theta\theta}(r, \theta = 0)\, \mathrm{d}r = \frac{2\, K_{\mathrm{IC}}}{\sqrt{2\pi\, d}} \, . \tag{14}$$

In the bimaterial body the value $\bar{\sigma}_{\rm crit}$ is related to the average stress $\bar{\sigma}(d,\alpha,\beta)$ calculated across a distance d in the material M2. Corresponding to (11) and (12) it is:

$$\bar{\sigma}(d,\alpha,\beta) = \frac{1}{d}\int\limits_0^d \sigma_{\theta\theta}(r,\theta=0)\,{\rm d}r = H_{\rm I}\,d^{-p}\,(2-p+g_{\rm R}) \ , \qquad (15)$$

where $g_{\rm R}$ is defined in (9). The average stress $\bar{\sigma}(d,\alpha,\beta)$ depends on the distance d and on the bimaterial composite parameters α, β (10), i.e. on the elastic mismatch of the materials.

The material ahead of the crack tip fractures when the value of the mean stress $\bar{\sigma}(d,\alpha,\beta)$ exceeds its critical value $\bar{\sigma}_{\rm crit}$ defined in (14):

$$\bar{\sigma}(d,\alpha,\beta) > \bar{\sigma}_{\rm crit} \qquad (16)$$

and the fracture condition has the form

$$H_{\rm I} > H_{\rm IC} \ , \qquad (17)$$

where $H_{\rm IC}$ is the critical value of the generalized stress intensity factor (the generalized fracture toughness). The value $H_{\rm IC}$ can be expressed as

$$H_{\rm IC} = K_{\rm IC}\,\frac{2\,d^{p-1/2}}{2-\lambda+g_{\rm R}} \ . \qquad (18)$$

The value of the critical stress $\bar{\sigma}_{\rm crit}$ depends on the fracture toughness $K_{\rm IC}$ of the material M2, on the elastic mismatch of both materials and on the value of d.

The choice of the dimension d in front of the crack tip depends on mechanisms of crack propagation. Usually, it is related to the grain size of the material M2, if the intergranular cracking is assumed.

3.1.1. The critical stress

The application of (18) makes it possible to estimate the critical applied stress $\sigma_{\rm crit}$. The critical stress is the value of the external applied tensile stress at the moment when the crack starts propagating from the bimaterial interface into the material M2 (we assume that for crack propagation the mode I is predominant)

$$\sigma_{\rm crit} = \frac{H_{\rm IC}\,\sigma_{\rm appl}}{H_{\rm I}(\sigma_{\rm appl})} \ . \qquad (19)$$

If the value of the applied stress $\sigma_{\rm appl} > \sigma_{\rm crit}$ the crack will grow into the material M2. The value of $\sigma_{\rm crit}$ depends on the geometry and the boundary conditions, on the composite parameters α and β, and on the fracture toughness $K_{\rm IC}$ of the material M2. Unlike the critical applied stress $\sigma_{\rm crit}$ in the case of a homogeneous material, it depends on the chosen distance d, see eq. (16).

3.2. Criterion based on a plastic zone size ahead of a crack tip

Typically of ductile materials, crack propagation is controlled by the plastic zone area $R_{\rm p}$ ahead of the crack tip (see Fig. 5). The fact that the plastic zone size is a controlling variable for fatigue crack propagation, e.g. [22–24] is the idea underlying the next criterion. The following criterion of stability can be used for estimation of the threshold stress value $\sigma_{\rm th}$ for a crack with its tip at the interface which is cyclically loaded.

For cyclic loading, the plastic zone forms ahead of the growing crack. Assuming small scale yielding conditions and a homogeneous body, there exists a single-valued relationship between the size of the plastic zone at the crack tip and the value of the corresponding stress intensity factor controlling the fatigue crack propagation rate. The behavior of a fatigue crack with its tip at an interface is then controlled by the plastic zone created in the material M2, see Fig. 5 and [25].

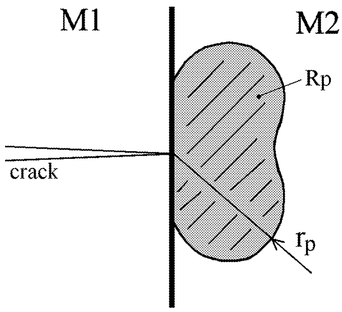

Fig.5: The plastic zone in the material M2 created by the crack with its tip at the boundary

In the following we assume that for fatigue crack propagation the controlling variable is the area of the plastic zone $R_{\rm p}$ around the tip of the crack, i.e. the variable $L = R_{\rm p}$.

The shape of the plastic zone predicted by LEFM for small scale yielding can be obtained by substituting the corresponding stresses into the Mises yield condition:

$$\sigma_{\rm ef} = \sigma_0 \ , \tag{20}$$

where $\sigma_{\rm ef}$ is the effective stress and σ_0 is the yield stress (for the material M2 in our case). For a homogeneous body and the loading level $K_{\rm I}$ (we assume a normal mode of loading) the value of the area $R_{\rm p}$ is given by (e.g. [22])

$$R_{\rm p} = \left(\frac{K_{\rm I}}{\sigma_0}\right)^4 f_{\rm hom}(\nu) \ , \tag{21}$$

$$f_{\rm hom}(\nu) = \frac{64(12\nu^4\pi - 24\nu^3\pi + 21\nu^2\pi - 9\nu\pi + 32\nu^4 - 64\nu^3 + 52\nu^2 - 20\nu + 3) + 123\pi}{1024\pi^2} \ .$$

In bi-material body the location of the curve corresponding to the elastic-plastic boundary in M2 for the crack with its tip at the boundary can be obtained by substituting the stress components (8) into the Mises condition (20). The radius of the plastic zone $r_{\rm P}$ is then

$$r_{\rm P} = \left(\frac{H_{\rm I}}{\sigma_0}\right)^{\frac{1}{p}} \left\{ \frac{(-p+1)^2}{2\pi} \left[6\, g_{\rm R}\, p \left(\sin(p\,\theta)\, \sin((-p-2)\,\theta) - \right.\right.\right.$$
$$\left.\left.\left. - \cos(p\,\theta)\, \cos((-p-2)\,\theta)\right) + 16\,\nu\, \cos(p\,\theta)^2 \left(\nu + \frac{1}{4\,\nu} - 1\right) + 3\,(g_{\rm R}^2 + p^2) \right] \right\}^{\frac{1}{2p}} \ . \tag{22}$$

By analogy to (21), in the case of a crack with its tip at the interface and for the loading level H_I, the plastic zone size can be written in the form

$$R_\mathrm{p} = \left(\frac{H_\mathrm{I}}{\sigma_0}\right)^{\frac{1}{p}} f(\alpha,\beta,\nu) \ , \tag{23}$$

where $f(\alpha,\beta,\nu)$ is a known function obtained by integration (22).

We assume that the areas of the plastic zones in homogeneous and bimaterial bodies, given by the equations (21) and (23) are the same, if $K_\mathrm{I} = K_\mathrm{th}$ (K_th is the threshold value of stress intensity factor) and $H_\mathrm{I} = H_\mathrm{th}$, i.e.,

$$R_\mathrm{p}(K_\mathrm{I} = K_\mathrm{th}) = R_\mathrm{p}(H_\mathrm{I} = H_\mathrm{th}) \ . \tag{24}$$

The relation $H_\mathrm{th} = H_\mathrm{th}(K_\mathrm{th},\alpha,\beta)$ can then be written in the form

$$H_\mathrm{th} = K_\mathrm{th}^{2p}\,\sigma_0^{(1-2p)} \left[\frac{f_\mathrm{hom}(\nu)}{f(\alpha,\beta,\nu)}\right]^{\frac{p}{2}} \ , \tag{25}$$

where σ_0 is the yield stress of the material M2.

If $H_\mathrm{I}(\sigma_\mathrm{appl}) < H_\mathrm{th}$, the rate of propagation of the fatigue crack will be zero, i.e. the fatigue crack will stop at the interface and will not propagate into the material M2. The condition $H_\mathrm{I}(\sigma_\mathrm{th}) = H_\mathrm{th}$ then makes it possible to determine the threshold applied stress σ_th as a function of the composite parameters α and β (10):

$$\sigma_\mathrm{th} = \frac{H_\mathrm{th}\,\sigma_\mathrm{appl}}{H_\mathrm{I}(\sigma_\mathrm{appl})} \ . \tag{26}$$

If the value of the applied stress is $\sigma_\mathrm{appl} < \sigma_\mathrm{th}$ then the fatigue crack will not propagate into the material M2.

3.3. Criterion based on the strain energy density factor

One of the basic criteria existing in LEFM is Sih's strain energy density concept [26], [27]. The strain energy density concept was originally proposed for a crack in homogeneous material.

The strain energy density w is given by:

$$\begin{aligned} w = \frac{\mathrm{d}W}{\mathrm{d}V} &= \frac{1}{8\,\mu}\left[k\,(\sigma_\mathrm{rr}+\sigma_{\theta\theta})^2 + (\sigma_\mathrm{rr}-\sigma_{\theta\theta})^2 + 4\,\sigma_{\mathrm{r}\theta}^2\right] = \\ &= \frac{1}{r}\left(a_{11}\,K_\mathrm{I}^2 + 2\,a_{12}\,K_\mathrm{I}\,K_\mathrm{II} + a_{22}\,K_\mathrm{II}^2\right) \ , \end{aligned} \tag{27}$$

where r is a distance from a crack tip, μ is a shear modulus, σ_rr, $\sigma_{\theta\theta}$ and $\sigma_{\mathrm{r}\theta}$ are polar stress components and the constant $k = (1-\nu)/(1+\nu)$ for plane stress and $k = (1-2\,\nu)$ for plane strain conditions. a_{11}, a_{12} and a_{22} are known functions of θ. Sih introduced the strain energy density factor S, which is independent of the distance r:

$$S = w\,r = a_{11}\,K_\mathrm{I}^2 + 2\,a_{12}\,K_\mathrm{I}\,K_\mathrm{II} + a_{22}\,K_\mathrm{II}^2 \ . \tag{28}$$

It is assumed that the crack starts to grow in the direction θ_{m} where S possesses the minimum, i.e.:

$$\left(\frac{\partial S}{\partial \theta}\right)_{\theta_{\mathrm{m}}} = 0 \; , \qquad \left(\frac{\partial^2 S}{\partial \theta^2}\right)_{\theta_{\mathrm{m}}} > 0 \; . \tag{29}$$

By substituting the stress components (6) into the relation (28) we can obtain a generalized form of the strain energy density factor $\Sigma(r)$ for bimaterial body, which contrary to S depends on the radial distance from the crack tip, $\Sigma = \Sigma(r,\theta)$:

$$\Sigma(r,\theta) = r^{1-2p_1} \, A_{11} \, H_1^2 + r^{1-p_1-p_2} \, 2 \, A_{12} \, H_1 \, H_2 + r^{1-2p_2} \, A_{22} \, H_2^2 \; , \tag{30}$$

where H_1, H_2 are generalized stress intensity factors and

$$\begin{aligned}
A_{11} &= \frac{1}{16\,\mu} \left[k \left(f_{\mathrm{rr}_1} + f_{\theta\theta_1} \right)^2 + \left(f_{\mathrm{rr}_1} - f_{\theta\theta_1} \right)^2 + 4 \, f_{\mathrm{r}\theta_1}^2 \right] \; , \\
A_{12} &= \frac{1}{16\,\mu} \left[(k+1) \left(f_{\mathrm{rr}_1} \, f_{\mathrm{rr}_2} + f_{\theta\theta_1} \, f_{\theta\theta_2} \right) + (k-1) \left(f_{\mathrm{rr}_1} \, f_{\theta\theta_2} + f_{\theta\theta_1} \, f_{\mathrm{rr}_2} \right) + 4 \, f_{\mathrm{r}\theta_1} \, f_{\mathrm{r}\theta_2} \right] \; , \\
A_{22} &= \frac{1}{16\,\mu} \left[k \left(f_{\mathrm{rr}_2} + f_{\theta\theta_2} \right)^2 + \left(f_{\mathrm{rr}_2} - f_{\theta\theta_2} \right)^2 + 4 \, f_{\mathrm{r}\theta_2}^2 \right] \; ,
\end{aligned}$$

where

$$\begin{aligned}
f_{\mathrm{rr}_1} &= -\lambda_1 \Big[a^{(1)} \, (\lambda_1 + 1) \, \sin(\lambda_1 + 1)\theta + b^{(1)} \, (\lambda_1 + 1) \, \cos(\lambda_1 + 1)\theta + \\
&\qquad + c^{(1)} \, (\lambda_1 - 3) \, \sin(\lambda_1 - 1)\theta + d^{(1)} \, (\lambda_1 - 3) \, \cos(\lambda_1 - 1)\theta \Big] \; , \\
f_{\mathrm{rr}_2} &= -\lambda_2 \Big[a^{(2)} \, (\lambda_2 + 1) \, \sin(\lambda_2 + 1)\theta + b^{(2)} \, (\lambda_2 + 1) \, \cos(\lambda_2 + 1)\theta + \\
&\qquad + c^{(2)} \, (\lambda_2 - 3) \, \sin(\lambda_2 - 1)\theta + d^{(2)} \, (\lambda_2 - 3) \, \cos(\lambda_2 - 1)\theta \Big] \; , \\
f_{\theta\theta_1} &= \lambda_1 \, (\lambda_1 + 1) \Big[a^{(1)} \, \sin(\lambda_1 + 1)\theta + b^{(1)} \, \cos(\lambda_1 + 1)\theta + \\
&\qquad + c^{(1)} \, \sin(\lambda_1 - 1)\theta + d^{(1)} \, \cos(\lambda_1 - 1)\theta \Big] \; , \\
f_{\theta\theta_2} &= \lambda_2 \, (\lambda_2 + 1) \Big[a^{(2)} \, \sin(\lambda_2 + 1)\theta + b^{(2)} \, \cos(\lambda_2 + 1)\theta + \\
&\qquad + c^{(2)} \, \sin(\lambda_2 - 1)\theta + d^{(2)} \, \cos(\lambda_2 - 1)\theta \Big] \; , \\
f_{\mathrm{r}\theta_1} &= -\lambda_1 \Big[a^{(1)} \, (\lambda_1 + 1) \, \cos(\lambda_1 + 1)\theta - b^{(1)} \, (\lambda_1 + 1) \, \sin(\lambda_1 + 1)\theta + \\
&\qquad + c^{(1)} \, (\lambda_1 - 1) \, \cos(\lambda_1 - 1)\theta - d^{(1)} \, (\lambda_1 - 1) \, \sin(\lambda_1 - 1)\theta \Big] \; , \\
f_{\mathrm{r}\theta_2} &= -\lambda_2 \Big[a^{(2)} \, (\lambda_2 + 1) \, \cos(\lambda_2 + 1)\theta - b^{(2)} \, (\lambda_2 + 1) \, \sin(\lambda_2 + 1)\theta + \\
&\qquad + c^{(2)} \, (\lambda_2 - 1) \, \cos(\lambda_2 - 1)\theta - d^{(2)} \, (\lambda_2 - 1) \, \sin(\lambda_2 - 1)\theta \Big] \; .
\end{aligned}$$

The formulation of the stability criterion derived from Sih's strain energy density factor for the general singular stress concentrator was introduced and in detail described in Part II of this work. In the following the strain energy density concept is used for the formulation of the stability criteria in the case of a crack generally oriented with the interface, where two singularities exist.

By substituting the expressions for the stress components (8) to the general expression for strain energy density (27) we can obtain the following final expression for the threshold value of the generalized stress intensity factor for the normal mode of loading:

$$H_{\mathrm{th}} = \left(\frac{1 - 2\,\nu}{(1-p)^2\,[4\,(1-2\,\nu) + (g_{\mathrm{R}} - p)^p]} \right)^{\frac{1}{2}} r^{p-\frac{1}{2}}\,K_{\mathrm{th}} \ . \tag{31}$$

The distance r at which the criterion is applied influences the resulting values of the threshold stress, but the dependence is not very strong. The distance r is related to the crack propagation mechanism. For a cleavage type of fracture, the distance r can be related to the grain size of the material or the size of the plastic zone in front of the crack tip in the case of crack propagation under the conditions of high cycle fatigue. The threshold stress σ_{th} can then be calculated from (26).

This criterion is able to be used with an advantage for mixed mode loading (e.g. a crack arbitrarily oriented to the bimaterial interface) in the case of brittle fracture or fatigue.

Existence of two singularities

If there are two different singularities (a crack arbitrarily oriented to the bimaterial interface, a bimaterial V-notch, etc.), it is possible to make full use of the stability criteria described above. In order to determine the critical stress, the following simplification can be applied. The expression for the stress field around the crack tip (4) may be formally written as:

$$\sigma_{ij} = H_1 \left[\frac{1}{\sqrt{2\pi}}\, r^{-p_1}\, f_{1ij}(\theta,\dots) + k\, \frac{1}{\sqrt{2\pi}}\, r^{-p_2}\, f_{2ij}(\theta,\dots) \right] \ , \tag{32}$$

where $k = H_2/H_1$. The values H_1 and H_2 determined from the numerical solution are both linearly dependent on the applied loading. This dependence is valid for the critical value of the applied stress as well and some of the introduced criteria can be used. The introduced stability criteria are advantageously usable for general stress concentrators, too.

Estimation of the crack propagation direction

The strain energy density approach can be used to estimate the direction in which the crack will propagate after passing through the bimaterial interface [17]. The crack will propagate into the material M2 in the direction of $\theta = \theta_{\mathrm{m}}$, which is related to the direction of the minimum of the strain energy density factor $\Sigma(r,\theta)$:

$$\Sigma(a_{\mathrm{o}}, \theta_{\mathrm{m}}) = \Sigma_{\min}(a_{\mathrm{o}}, \theta) \ . \tag{33}$$

The value $r = a_{\mathrm{o}}$ depends on the crack propagation mechanism and can be chosen, e.g. like corresponding to the plastic zone size ahead of the crack tip [28]. It is assumed that the next behaviour of the crack is controlled by the rules of LEFM. The unknown angle θ_{m} can be obtained from the solution of the relationship:

$$\left(\frac{\partial \Sigma}{\partial \theta} \right)_{\theta_{\mathrm{m}}} = 0 \ , \qquad \left(\frac{\partial^2 \Sigma}{\partial \theta^2} \right)_{\theta_{\mathrm{m}}} > 0 \ . \tag{34}$$

The equation (30) for the generalized strain energy density factor can be rewritten as:

$$\Sigma = H_1^2 \left[r^{1-2p_1}\, A_{11} + 2\, r^{1-p_1-p_2}\, A_{12}\, \frac{H_2}{H_1} + r^{1-2p_2}\, A_{22} \left(\frac{H_2}{H_1} \right)^2 \right] \ . \tag{35}$$

The values H_1 and H_2 are independent of the polar coordinate θ. The condition for the crack propagation direction has the form:

$$\left[r^{1-2p_1} \frac{\partial A_{11}}{\partial \theta} + 2\, r^{1-p_1-p_2} \frac{H_2}{H_1} \frac{\partial A_{12}}{\partial \theta} + r^{1-2p_2} \left(\frac{H_2}{H_1} \right)^2 \frac{\partial A_{22}}{\partial \theta} \right]_{\theta_m} = 0 \; . \tag{36}$$

It is evident that the crack propagation direction does not depend on the absolute values of H_1 and H_2.

4. Numerical examples

4.1. Estimation of the threshold stresses

As an example of the approaches presented, the results concerning a fatigue crack growing perpendicularly in a protective coating and penetrating through the interface and into the substrate material are presented and a comparison of the suggested criteria is made in the following.

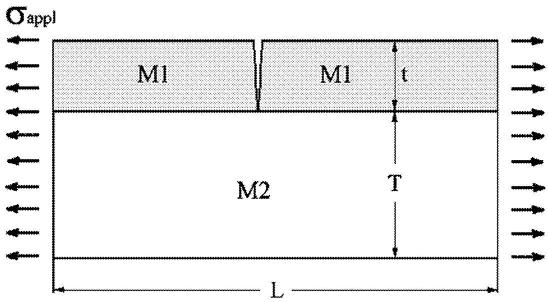

Fig.6: The bimaterial body with an edge crack under tensile loading considered in the numerical example; $T = 25$ mm, $t = 12.5$ mm, $L = 75$ mm

For comparison a bi-material sample loaded by tensile stress was chosen (see Fig. 6). In this case the crack is loaded by the normal mode and there is only one stress singularity. The calculations were performed for different ratios of Young's modulus (E_1/E_2) in a wide range from 0.2 to 5. Poisson's ratios were the same in all cases $\nu_1 = \nu_2 = 0.3$. For the parametric study the following constants were used: $d = 0.01$ mm (in eq. (18)), $\sigma_0 = 600$ MPa (in eq. (25)), $r = 0.001$ mm (in eq. (31)) and $K_{\rm th} = 5\,{\rm MPa\,m}^{1/2}$. For the computation the finite element system Ansys [29] was used. The threshold stress $\sigma_{\rm th}$ was estimated by means of the equation (26). The normalized results are shown in Fig. 8. For the normalization the value of the fatigue threshold stress determined for the configuration according Fig. 6 and for homogeneous material M2 was used.

From Fig. 7 it is evident that all criteria provide similar results of the threshold stress and are in a good agreement. The results presented show the dependence of the threshold stress on the ratio E_2/E_1 of Young's modulus and indicate a relatively strong decrease in the threshold stress in the cases when Young's modulus of the material M1, E_1, is larger than Young's modulus of the material M2, E_2. For the opposite ratio of Young's moduli ($E_2 > E_1$) the critical stress increases.

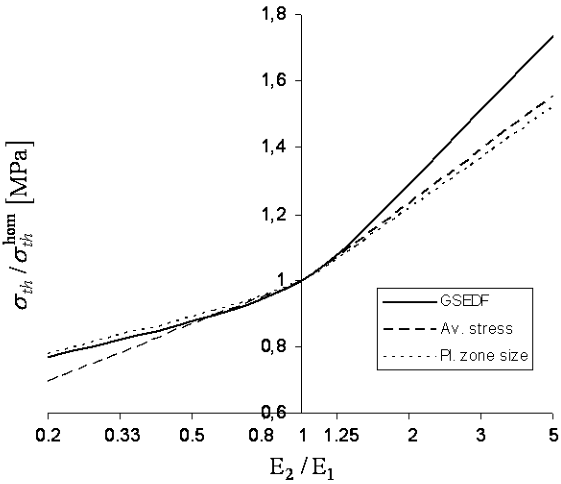

Fig.7: The comparison of the calculated threshold stresses (normalized by threshold stress of homogeneous material M2) from the criteria based on : the generalized strain energy density factor (GSEDF), the value of the average stress ahead of a crack tip (av. stress) and the area of the plastic zone in front of a crack tip (pl. zone size)

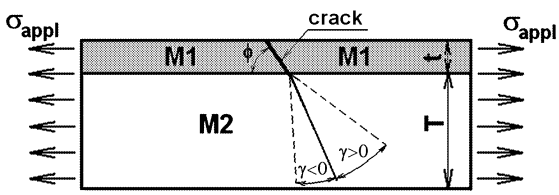

Fig.8: The model of a bimaterial body with applied loading used for FE calculations of changes of the crack propagation direction after the crack passed the interface; $\gamma = 0$ [deg] means propagation direction in homogeneous material; $T = 15$ mm, $t = 1$ mm, $L = 60$ mm

ϕ [deg]	90	80	70	60	50
$E_1/E_2 = 2$	0	−6	−11	−16	−19
$E_1/E_2 = 0.5$	0	4	8	10	11

Tab.1: The resulting deviation ϕ [deg] of the crack propagation direction after the crack passed the bimaterial interface in comparison with propagation in homogeneous material ($E_1/E_2 = 1$); the value ϕ represents the angle between the crack and the interface (Fig. 8)

4.2. Crack propagation direction

A bimaterial body loaded by tensile stress was chosen as a numerical example of estimation of the crack propagation direction after the crack passed through the interface, see Fig. 8. The applied stress $\sigma_{\rm appl} = 100$ MPa and $\nu_1 = \nu_2 = 0.3$, $E_2 = 2\times10^5$ MPa, two different values of E_1 ($E_1 = 1\times10^5$ and $E_1 = 4\times10^5$ MPa), the threshold value of the stress intensity factor of the material M2 $K_{\rm th} = 6.42$ MPa m$^{1/2}$ and the yield stress of the material M2 $\sigma_0 = 800$ MPa were used for numerical calculations.

Numerical calculations were performed for five different angles ϕ between the crack and the interface. For estimation of the crack propagation direction the equation (36) was used. The corresponding results are summarized in Table 1. See [17] for details.

From Table 1 it is evident that for $E_1/E_2 > 1$ the crack inclines to the direction normal to the bimaterial interface within its next propagation. For ratio $E_1/E_2 < 1$ the crack deviates from direction normal to bimaterial interface. More details and a discussion have been published in [17].

5. Discussion

The stability criteria presented for a crack with its tip at the interface are based on three different physical principles:

- on the magnitude of the average stress ahead of the crack tip
- on the plastic zone size ahead of the crack tip
- on the energy approach introduced by Sih based on the generalized strain energy density factor

Each of the described criteria has a different fundamental principle and is useable for different crack propagation mechanism. The results calculated by means of all criteria are in a good agreement in a wide range of Young's modulus ratio, see Fig. 8. It can be concluded that for the current materials for which the ratio E_1/E_2 is from the interval $0.2 < E_1/E_2 < 5$ all approaches give comparable results. All criteria are linked together by the conditions of LEFM.

The advantage of the presented criteria is the fact that no new material characteristics are needed and no new measurements on bimaterial bodies are necessary for estimation of the critical values for crack propagation. For estimation of the critical stress of crack propagation through the bimaterial interface it is enough to know the critical value of the fracture toughness (K_{IC}) of the material M2 only.

The proper values of the parameters r (in eq. (31)) or d (in eq. (18)) have to be chosen, but their influence on the resulting critical stress is not strong. These parameters allow to include the real material structure and the mechanisms of crack propagation to the estimation of the critical stress.

6. Conclusions

Three criteria dealing with the propagation of a crack through the interface between two materials are introduced in the paper. Attention is paid to the case of a crack touching the interface. These procedures make it possible to quantify the effect of the interface on the critical value at which a fatigue crack propagates from the first material into the second one. The procedures are based on the extension of linear elastic fracture mechanics to general singular stress concentrators.

It follows from the results presented that the corresponding critical value can be strongly influenced by the existence of an interface between two materials.

Basically, if a crack propagates from a stiffer material into a softer one (i.e. $E_1/E_2 > 1$), the value of the critical stress σ_{crit} (in case of brittle materials M1 and M2) or the threshold stress σ_{th} decreases. In the opposite case (for $E_1/E_2 < 1$) the critical stress for crack propagation σ_{crit} (or the threshold stress σ_{th}) increases.

From a practical point of view, the suggested procedures are general and can be used in considerations on the crack stability in composite materials, protective layers, etc. The results obtained contribute to a better understanding of damage caused by cracks in the above mentioned structures.

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