CALCULATION OF NATURAL VIBRATION OF LINEAR UNDAMPED ROTATIONALLY PERIODIC STRUCTURES

Pavel Polach*

The article presents the method of the calculation of natural frequencies and mode shapes of the linear undamped rotationally periodic systems. The method is applied in the calculation of the natural vibration characteristics of the steam turbine bladed disks.

Key words: *rotational periodicity, subsystem, natural vibration, bladed disk, steam turbine*

1. Motivation

The approach towards the calculation of the natural vibration characteristics (i.e. natural frequencies and mode shapes) of steam turbine bladed disks using the rotational periodicity of the structure was solved in [1]. The verification of the method on the tested examples [2] and the calculation of the natural vibration characteristics of steam turbine bladed disk with blades of the 220Z-1085 type connected with the continuous binding [3], [4], in which it was possible to compare the results of the calculations with the experimental measurement evaluation [5], have been published in scientific conferences so far. The motivation for writing papers [6], [7] and this article was the topical application of the method in calculating the natural vibration characteristics of the steam turbine bladed disk with the blades of the ZN340-2 type with the continuous binding by shrouding and tie-boss [8], real possibility of further application of the method when developing the steam turbine bladed disks of a new design (in connection with the supposed development of power engineering in the Czech Republic), reminding the professional public of the possibility of a broader utilization of the problematics that had not been developed in the Czech Republic for several years and discussion concerning the prospects of further possible development of the methods that use the rotational periodicity of the systems for the calculation of dynamic properties of bladed disks of rotating machines.

2. Periodic structures

Description of various methods that use specific properties of the structure for the calculation of its investigated characteristics or behaviour under the given conditions can be found in the appropriate literature relatively often. The periodicity of the system is used to solve various problems. The origin of the papers dates back mainly to the years after 1970, when thanks to the development of computer technology it was possible to use effectively the derived procedures. Due to a large amount of publications dealing with using periodicity

* Dr. Ing. P. Polach, Section of Materials and Mechanical Engineering Research, ŠKODA VÝZKUM s.r.o., Tylova 1/57, 316 00 Plzeň

of the system for the calculation of its dynamic properties and due to the fact that most of them present the solution of a narrow range of problems of other fields than dynamic properties of rotationally periodic systems, only an informative survey of the problematics is given (its description in more detail is presented e.g. in [1], [9] and [10]).

In the theoretical works some authors deal with processes in infinite or semi-infinite open (linear) systems. Both infinite and semi-infinite open periodic systems are composed of the infinite amount of subsystems; at the infinite system it is possible to assess neither its beginning nor its end, at the semi-infinite system it is possible to assess its beginning. Finite systems can be generally divided into open ones (boundary conditions must be considered) and cyclic (closed) ones. Some works deal with the influence of slight mutual deviations of the subsystems on the change in the whole system behaviour (so called nearly-periodic systems are concerned).

There are two basic approaches to solving the dynamic behaviour of periodic systems: a vibration one and a wave one (e.g. [9], [11]). The wave approach is applied in the investigation of the wave propagation through the periodic system but it can be used for obtaining information concerning the periodic system vibration. Similarly in the vibration approach, in which the output of the investigation should be the periodic system vibration, the quantities connected with the wave propagation can be determined using the investigation results.

From the point of view of generality, type of solved physical models, and boundary conditions the publications by Vladimír Pětrovský are unique and very important for the investigation of the natural vibration characteristics of the periodic undamped systems. They deal with cyclic systems [12], finite open systems [13] and symmetric periodic systems [14]. In [12], [13] and [14] numerical applications of the approaches are not presented on concrete examples. The works by Pětrovský are just known to the limited number of the specialists involved in the problems of periodic systems because they were published only as the internal reports of ÚTSSK Plzeň or ŠKODA VÝZKUM s.r.o.

Some publications deal exclusively with problems of rotationally periodic systems. E.g. Kempner and Efremova determine, using the transfer matrix method, natural frequencies of the steam turbine blades with the continuous binding [15] (a blade with the appropriate part of shrouding is considered to be a subsystem). Fricker and Potter investigate the response of rotationally periodic structures on transient forces and prescribed motions. The response is obtained by the solution of the equation of motion of a single substructure for a number of different spatial Fourier harmonic of force [16]. Problems of the natural vibration in [17] are solved in a similar way. The author considers a case in which the boundary conditions of cyclosymmetry are not fulfilled and solves the reduction of the number of degrees of freedom by means of so called spectral condensation (the method derived by the author is concerned [17]). In [18] DiMaggio, Duron and Davis deal with the experimental identification of the mode shapes of the turbopump bladed disk. A helicopter propeller is a rotationally periodic structure as well – its natural frequencies and mode shapes are investigated by Dubigeon and Michon [19]. As to the Czech authors involved in the problems of rotationally periodic structures Ivan Krásný must be mentioned. In [20] he uses the method for the static solution of rotationally periodic structures subjected to general loads, and calculates the natural vibration characteristics of undamped systems. The analyses of computational time demands and solution errors are also included in this paper. In [21] he solves the reduction of degrees of freedom of a subsystem discretized using FEM, the influence of the reduction of degrees of freedom on the computing time of natural frequencies and on the accuracy of results. In [22] (together with Mančal) he gives in words the succession of operations when calculating the natural frequencies of the steam turbine bladed disks and the compressor impellers. The contribution by Burda [23], which follows from [20], also deals with the problems of the natural vibration of bladed disks.

3. Steam turbine bladed disks and the framework description of a computational model

As mentioned in the article introduction the main reason for the derivation of the calculation method for the natural vibration characteristics of rotationally periodic systems was especially its usability at designing the structural solution of the steam turbine bladed disks.

When designing turbines the producer must take into account many sometimes even opposite points of view. The customer is interested, besides the price, efficiency and lifetime, especially in the reliability. In order to reduce the probability of the occurrence of unexpected failures the producer must consider many data when designing the turbine and its parts, which, after the evaluation, give information about the suitability of the turbine design for the given operational condition. From the point of view of dynamics, the knowledge of the natural vibration characteristics of the turbine individual structural parts is important for the assessment of the design suitability. On the basis of those characteristics it is possible to assess according to various criteria and experience if, due to acting forces, the danger of resonant vibration excitation does not threaten.

In the computational model for the determination of the natural vibration characteristics based on the rotational periodicity of the bladed disk it is considered that the bladed disk can be divided into the certain number of identical parts – subsystems. It is advisable to choose the subsystem to be formed by a disk sector with one blade (i.e. the smallest possible identical part). The subsystem discretization will be performed in such a way that the subsystem may be coupled in an equal number of points in the same degrees of freedom to their left-side and right-side adjacent subsystems. After the mathematical formulation of the problem it is possible to derive relations for the calculation of natural frequencies and mode shapes of the whole system (i.e. the bladed disk) [1] using the theory of the solution of the matrix difference equations [24]. Compared with the solution of the system as a whole the main advantage of the mentioned method (using the rotational periodicity) consists in the fact that the order of stiffness and mass matrices does not increase (the need of their assembling follows from the process of solution) but it is at most double than in case of considering a single subsystem. Thus the solution is less demanding for the computer operating memory and computing time. Performing a lower number of numerical operations and thus reducing the probability of a computing error is the result of a lower number of solved equations.

When applying the natural vibration characteristics calculating method to the steam turbine bladed disks, the subsystem stiffness and mass matrices (the whole system is considered linear and undamped) are assembled using the finite element method (using the COSMOS/M software). The problem formulation leads to the solution of the generalized eigenvalue problem and the natural vibration characteristics are calculated using the subspace iteration method. A special in-house software was created to visualize the mode shapes. The applicability of the method was verified when determining the natural vibration characteristics of the steam turbine bladed disk with the blades of the 220Z-1085 type connected by the continuous binding [1], [3].

4. Relations for the calculation of the natural vibration characteristics of the linear undamped rotationally periodic system

Detailed derivation of the mathematical relations for the calculation of the natural vibration characteristics of the linear undamped rotationally periodic system is performed in [1]. In contradiction to [1] the possibility of the elimination of the subsystem internal degrees of freedom is not presented in this article.

Let the finite periodic system be composed of the definite number M of identical parts – subsystems. The subsystem discretization will be performed in such a way, that it may be coupled in identical number N of points in identical degrees of freedom to their left-side and right-side adjacent subsystems (see Fig. 1). There are no requirements imposed on internal points of the subsystem.

Mathematically, this approach to the problem formulation leads to assembling and solving the matrix difference equations.

By introducing the condition that the whole system is linear and the motion of the k-th subsystem is investigated during its harmonic vibration it is possible to formulate the displacement vector $\mathbf{u}_k(t)$ of the k-th subsystem in the form

$$\mathbf{u}_k(t) = \mathbf{U}_k \, \mathrm{e}^{\mathrm{i}\,\omega\,t} \tag{1}$$

where $\mathbf{U}_k$ is the vector of displacement amplitudes of the k-th subsystem, ω is the angular frequency, t is the time, i is the imaginary unit. The vector of generalized forces $\mathbf{q}_k(t)$ acting on the k-th subsystem can be formulated in the form

$$\mathbf{q}_k(t) = \mathbf{Q}_k \, \mathrm{e}^{\mathrm{i}\,\omega\,t} \tag{2}$$

where $\mathbf{Q}_k$ is the vector of amplitudes of generalized forces $\mathbf{q}_k(t)$ acting on the k-th subsystem.

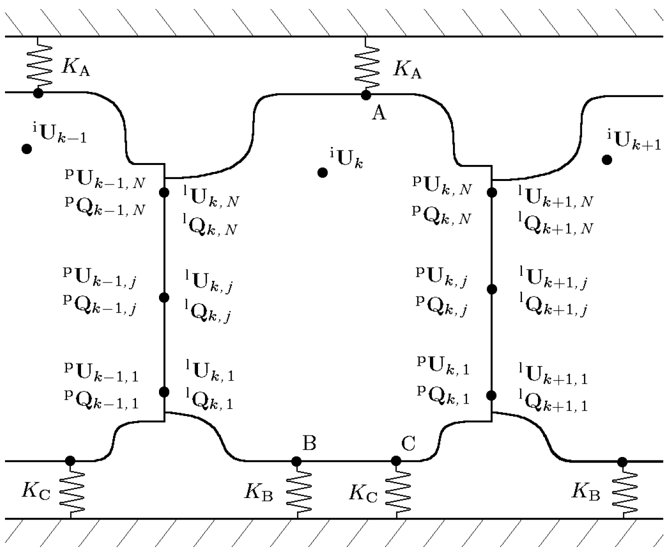

Fig.1: The k-th subsystem and its couplings with the adjacent subsystems

Thus it is supposed that the vector of generalized forces $\mathbf{q}_k(t)$ is proportional to the instantaneous state of the subsystem and it changes proportionally to its displacements. The relation between the amplitude of the subsystem displacements and the amplitude of generalized forces acting on it is described in the matrix equation

$$\mathbf{D}(\omega)\,\mathbf{U}_k = \mathbf{Q}_k \tag{3}$$

where $\mathbf{D}(\omega)$ is the frequency dependent dynamic stiffness matrix identical for all subsystems $(k = 1, 2, \ldots, M)$.

The vector of displacements amplitudes $\mathbf{U}_k$ of the k-th subsystem can be partitioned into subvectors corresponding to the degrees of freedom ${}^{\mathrm{l}}\mathbf{U}_k$, in which the k-th subsystem is coupled with the subsystem $k-1$, to the degrees of freedom ${}^{\mathrm{p}}\mathbf{U}_k$, in which it is coupled with the subsystem $k+1$ and to the internal degrees of freedom ${}^{\mathrm{i}}\mathbf{U}_k$ (see Fig. 1). The vector of the amplitudes of generalized forces $\mathbf{Q}_k$ can be partitioned in the same way (see Fig. 1).

In the points of coupling the k-th subsystem with the adjacent subsystems the compatibility conditions hold

$$\begin{aligned} {}^{\mathrm{p}}\mathbf{U}_{k-1} &= {}^{\mathrm{l}}\mathbf{U}_k \ , \\ {}^{\mathrm{p}}\mathbf{U}_k &= {}^{\mathrm{l}}\mathbf{U}_{k+1} \end{aligned} \tag{4}$$

and the conditions of equilibrium of the generalized coupling forces hold

$$\begin{aligned} {}^{\mathrm{p}}\mathbf{Q}_{k-1} &= -{}^{\mathrm{l}}\mathbf{Q}_k \ , \\ {}^{\mathrm{p}}\mathbf{Q}_k &= -{}^{\mathrm{l}}\mathbf{Q}_{k+1} \ . \end{aligned} \tag{5}$$

The dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem can be written using the submatrices corresponding to the individual groups of degrees of freedom. When introducing the condition that the generalized forces act on the subsystem only in the points common with the adjacent subsystems and that they do not act on the subsystem internal points, the matrix equation (3) can be written in the form

$$\begin{bmatrix} {}^{\mathrm{ll}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} & {}^{\mathrm{lp}}\mathbf{D} \\ {}^{\mathrm{il}}\mathbf{D} & {}^{\mathrm{ii}}\mathbf{D} & {}^{\mathrm{ip}}\mathbf{D} \\ {}^{\mathrm{pl}}\mathbf{D} & {}^{\mathrm{pi}}\mathbf{D} & {}^{\mathrm{pp}}\mathbf{D} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{l}}\mathbf{U}_k \\ {}^{\mathrm{i}}\mathbf{U}_k \\ {}^{\mathrm{p}}\mathbf{U}_k \end{bmatrix} = \begin{bmatrix} {}^{\mathrm{l}}\mathbf{Q}_k \\ \mathbf{0} \\ {}^{\mathrm{p}}\mathbf{Q}_k \end{bmatrix} \ . \tag{6}$$

Note: If the subsystems were coupled to the inertial frame, as it is illustrated in Fig. 1, by springs of stiffness K_{A}, K_{B} and K_{C} in nodes A, B and C these stiffnesses would be included in the submatrices ${}^{\mathrm{li}}\mathbf{D}$, ${}^{\mathrm{pi}}\mathbf{D}$ or ${}^{\mathrm{ii}}\mathbf{D}$ of the dynamic stiffness matrix $\mathbf{D}(\omega)$.

The equation obtained by multiplying the first row of the matrix equation (6) and transcribed for the subsystem $k+1$ is, after introducing the second compatibility condition (4) and the second condition of generalized coupling forces equilibrium (5), of the form

$${}^{\mathrm{ll}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_k + {}^{\mathrm{li}}\mathbf{D}\,{}^{\mathrm{i}}\mathbf{U}_{k+1} + {}^{\mathrm{lp}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_{k+1} = -{}^{\mathrm{p}}\mathbf{Q}_k \ . \tag{7}$$

Using the first compatibility condition (4) the following relations are obtained from the second and the third row of the matrix equation (6):

$${}^{\mathrm{il}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_{k-1} + {}^{\mathrm{ii}}\mathbf{D}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{ip}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_k = \mathbf{0} \ , \tag{8}$$

$${}^{\mathrm{pl}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_{k-1} + {}^{\mathrm{pi}}\mathbf{D}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{pp}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_k = {}^{\mathrm{p}}\mathbf{Q}_k \ . \tag{9}$$

Equations (7), (8) and (9) are the initial difference equations for the calculation of the subvectors of displacements amplitudes ${}^{\mathrm{p}}\mathbf{U}_k$ and ${}^{\mathrm{i}}\mathbf{U}_k$ and the subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{p}}\mathbf{Q}_k$.

The more suitable procedure of the two possible ones (mentioned in [1]), which can be used for the composition of relations for the calculation of natural frequencies and mode shapes of the periodic system, leads to the solution of the system of homogeneous linear difference equations of the second type (the variable in these equations is k; $k = 1, 2, \ldots, M$) Using this procedure the subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{p}}\mathbf{Q}_k$ is eliminated from the difference equations (7), (8) and (9) and only the subvectors of displacements amplitudes ${}^{\mathrm{p}}\mathbf{U}_k$ and ${}^{\mathrm{i}}\mathbf{U}_k$ are determined.

The subvector of the amplitudes of generalized coupling forces ${}^{\mathrm{p}}\mathbf{Q}_k$ is eliminated by the summation of the equations (7) and (9):

$$ {}^{\mathrm{pl}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_{k-1} + ({}^{\mathrm{ll}}\mathbf{D} + {}^{\mathrm{pp}}\mathbf{D})\,{}^{\mathrm{p}}\mathbf{U}_k + {}^{\mathrm{pi}}\mathbf{D}\,{}^{\mathrm{i}}\mathbf{U}_k + {}^{\mathrm{lp}}\mathbf{D}\,{}^{\mathrm{p}}\mathbf{U}_{k+1} + {}^{\mathrm{li}}\mathbf{D}\,{}^{\mathrm{i}}\mathbf{U}_{k+1} = \mathbf{0}\ . \tag{10} $$

Matrix equations (8) and (10) can be expressed using one matrix equation

$$ \mathbf{E}\,{}^{\mathrm{pi}}\mathbf{U}_{k-1} + \mathbf{F}\,{}^{\mathrm{pi}}\mathbf{U}_k + \mathbf{G}\,{}^{\mathrm{pi}}\mathbf{U}_{k+1} = \mathbf{0} \tag{11} $$

where

$$ \mathbf{E} = \begin{bmatrix} {}^{\mathrm{pl}}\mathbf{D} & \mathbf{0} \\ {}^{\mathrm{il}}\mathbf{D} & \mathbf{0} \end{bmatrix}\ , \qquad \mathbf{F} = \begin{bmatrix} {}^{\mathrm{ll}}\mathbf{D} + {}^{\mathrm{pp}}\mathbf{D} & {}^{\mathrm{pi}}\mathbf{D} \\ {}^{\mathrm{ip}}\mathbf{D} & {}^{\mathrm{ii}}\mathbf{D} \end{bmatrix} \qquad \text{and} \qquad \mathbf{G} = \begin{bmatrix} {}^{\mathrm{lp}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\ . $$

In the process of solution (see Appendix A) the condition that the periodic system is undamped is introduced and cyclic condition:

$$ {}^{\mathrm{pi}}\mathbf{U}_{k+M} = {}^{\mathrm{pi}}\mathbf{U}_k\ , \quad \text{where} \quad k = 1, 2, \ldots, M \tag{12} $$

is used.

As the periodic system is considered undamped the dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem can be put in the form (see Appendix A):

$$ \mathbf{D}(\omega) = \mathbf{K} - \omega^2\,\mathbf{M} \tag{13} $$

where $\mathbf{K}$ is the stiffness matrix of the subsystem and $\mathbf{M}$ is the mass matrix of the subsystem.

After solving the matrix difference equations (see Appendix A; the course of solving complies with the conditions given in [24]) the vector of displacements amplitudes ${}^{\mathrm{pi}}\mathbf{U}_k$ is dependent on the optional parameter β for cyclic periodic systems (for the mode shapes of the rotationally periodic systems the parameter β means the number of nodal diameters). Two forms of equations are obtained for the calculation of natural frequencies and mode shapes of the whole periodic system in dependence on the parameter β value: the first form of equations (see relations from (14) to (19)) holds for $\beta = 0$ and in addition in case of even M $\beta = M/2$, the second form of equations (see relations from (20) to (25)) holds for $\beta = 1, 2, \ldots, M-1$ and when M is even with the condition $\beta \neq M/2$.

For $\beta = 0$ (i.e. $\alpha = 0$) and in addition in case of even M $\beta = M/2$ (i.e. $\alpha = \pi$), the solution of the characteristic equation (A4) (see Appendix A) will be one real root and for ${}^{\mathrm{pi}}\mathbf{U}_k$ relation (A23) (see Appendix A) will yield:

$$ {}^{\mathrm{pi}}_{\ \beta}\mathbf{U}_k = C\,\cos({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a} \tag{14} $$

where ${}_{\beta}\alpha = 2\pi\,\beta/M$, $\beta = 0$ and in addition in case of even M $\beta = M/2$.

When $\beta = 1, 2, \ldots, M-1$ and when M is even with the condition $\beta \neq M/2$ the solution of the characteristic equations (A4) (see Appendix A) for the rotationally periodic systems will be two complex conjugate roots and for ${}^{\mathrm{pi}}\mathbf{U}_k$ relation (A23) (see Appendix A) gives

$$ {}^{\mathrm{pi}}_{\beta}\mathbf{U}_k = C_1\,[\cos({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a} + \sin({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{b}] + C_2\,[\cos({}_{\beta}\alpha\,k)\,(-{}_{\beta}\mathbf{b}) + \sin({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a}] \tag{15} $$

where ${}_{\beta}\alpha = 2\pi\,\beta/M$, $\beta = 1, 2, \ldots, M-1$; when M is even with the condition $\beta \neq M/2$.

Characteristic vectors ${}_{\beta}\mathbf{a}$ and ${}_{\beta}\mathbf{b}$ are determined (for specifically chosen β) by substituting relations (14) or (15) into equation (11). Cyclic condition is fulfilled for the arbitrary value of the constant C in equation (14) or the constants C_1 and C_2 in equation (15).

Now let be considered the case in which characteristic equation (A4) (see Appendix A) has a real root. After substituting relation (14) into equation (11) it is obtained:

$$ \mathbf{E}\,C\,\cos[{}_{\beta}\alpha\,(k-1)]\,{}_{\beta}\mathbf{a} + \mathbf{F}\,C\,\cos({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a} + \mathbf{G}\,C\,\cos[{}_{\beta}\alpha\,(k+1)]\,{}_{\beta}\mathbf{a} = \mathbf{0} \tag{16} $$

where $\beta = 0$ and in addition in case of even M $\beta = M/2$.

Expressions $\cos[{}_{\beta}\alpha\,(k-1)]$ and $\cos[{}_{\beta}\alpha\,(k+1)]$ will be substituted by the expressions obtained according to the angle difference identities in equation (16) – see Appendix B.

Modifying equation (16) it is obtained (assumption $C \neq 0$):

$$ [\mathbf{F} + (\mathbf{E} + \mathbf{G})\,\cos{}_{\beta}\alpha]\,{}_{\beta}\mathbf{a} = \mathbf{0} \tag{17} $$

where $\beta = 0$ and in addition in case of even M $\beta = M/2$.

Following designation is performed:

$$ \begin{aligned} {}_{\beta}\mathbf{H} = {}_{\beta}\mathbf{H}_1 &= [\mathbf{F} + (\mathbf{E} + \mathbf{G})\,\cos{}_{\beta}\alpha] = \\ &= \begin{bmatrix} {}^{\mathrm{ll}}\mathbf{D} + {}^{\mathrm{pp}}\mathbf{D} & {}^{\mathrm{pi}}\mathbf{D} \\ {}^{\mathrm{ip}}\mathbf{D} & {}^{\mathrm{ii}}\mathbf{D} \end{bmatrix} + \left(\begin{bmatrix} {}^{\mathrm{pl}}\mathbf{D} & \mathbf{0} \\ {}^{\mathrm{il}}\mathbf{D} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^{\mathrm{lp}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \cos{}_{\beta}\alpha\ . \end{aligned} \tag{18} $$

The characteristic vector ${}_{\beta}\mathbf{a}$ can be determined from equation (17) and natural frequencies of the whole periodic system can be determined from the condition of a nontrivial solution:

$$ \det{}_{\beta}\mathbf{H} = \det{}_{\beta}\mathbf{H}_1 = 0\ . \tag{19} $$

Mode shapes of the whole periodic system can be calculated by substituting the characteristic vectors ${}_{\beta}\mathbf{a}$ into equation (14).

If two complex conjugate roots are the solution of the characteristic equation (A4) (see Appendix A), each particular solution separately is introduced into equation (11) (each one fulfils independently equation (11)) from relation (15) and two conditions for the calculation of ${}_{\beta}\mathbf{a}$ and ${}_{\beta}\mathbf{b}$ are obtained:

$$ \begin{aligned} &\mathbf{E}\,C_1\,\{\cos[{}_{\beta}\alpha\,(k-1)]\,{}_{\beta}\mathbf{a} + \sin[{}_{\beta}\alpha\,(k-1)]\,{}_{\beta}\mathbf{b}\} + \\ &\quad + \mathbf{F}\,C_1\,\{\cos({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a} + \sin({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{b}\} + \\ &\quad\quad + \mathbf{G}\,C_1\,\{\cos[{}_{\beta}\alpha\,(k+1)]\,{}_{\beta}\mathbf{a} + \sin[{}_{\beta}\alpha\,(k+1)]\,{}_{\beta}\mathbf{b}\} = \mathbf{0}\ , \\ &\mathbf{E}\,C_2\,\{\cos[{}_{\beta}\alpha\,(k-1)]\,(-{}_{\beta}\mathbf{b}) + \sin[{}_{\beta}\alpha\,(k-1)]\,{}_{\beta}\mathbf{a}\} + \\ &\quad + \mathbf{F}\,C_2\,\{\cos({}_{\beta}\alpha\,k)\,(-{}_{\beta}\mathbf{b}) + \sin({}_{\beta}\alpha\,k)\,{}_{\beta}\mathbf{a}\} + \\ &\quad\quad + \mathbf{G}\,C_2\,\{\cos[{}_{\beta}\alpha\,(k+1)]\,(-{}_{\beta}\mathbf{b}) + \sin[{}_{\beta}\alpha\,(k+1)]\,{}_{\beta}\mathbf{a}\} = \mathbf{0} \end{aligned} \tag{20} $$

where $\beta = 1, 2, \ldots, M-1$; when M is even with the condition $\beta \neq M/2$.

Equations (20) for the calculation of ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ must be fulfilled for each k, that is why $k = M$ can be selected without the loss of generality. After the modifications it is obtained (assumption $C_1 \neq 0$ and at the same time $C_2 \neq 0$):

$$\begin{aligned} &[\mathbf{F} + (\mathbf{E} + \mathbf{G})\cos{}_\beta\alpha]\,{}_\beta\mathbf{a} + (\mathbf{G} - \mathbf{E})\sin{}_\beta\alpha\,{}_\beta\mathbf{b} = \mathbf{0}\ , \\ &-(\mathbf{G} - \mathbf{E})\sin{}_\beta\alpha\,{}_\beta\mathbf{a} + +[\mathbf{F} + (\mathbf{E} + \mathbf{G})\cos{}_\beta\alpha]\,{}_\beta\mathbf{b} = \mathbf{0} \end{aligned} \tag{21}$$

where $\beta = 1, 2, \ldots, M - 1$; when M is even with the condition $\beta \neq M/2$.

Equations (21) can be written into one matrix equation:

$${}_\beta\mathbf{H}\,{}_\beta\mathbf{v} = \begin{bmatrix} {}_\beta\mathbf{H}_1 & {}_\beta\mathbf{H}_2 \\ -{}_\beta\mathbf{H}_2 & {}_\beta\mathbf{H}_1 \end{bmatrix} \begin{bmatrix} {}_\beta\mathbf{a} \\ {}_\beta\mathbf{b} \end{bmatrix} = \mathbf{0}\ . \tag{22}$$

Square submatrices ${}_\beta\mathbf{H}_1$ and ${}_\beta\mathbf{H}_2$ are of the n-order (as it has been already stated: n is dimension of the vector ${}^{\mathrm{pi}}\mathbf{U}_k$, or the order of the matrices $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{G}$).

Matrix ${}_\beta\mathbf{H}_1$ can be calculated in the same way as for the real root of the characteristic equation (18):

$$\begin{aligned} {}_\beta\mathbf{H}_1 &= [\mathbf{F} + (\mathbf{E} + \mathbf{G})\cos{}_\beta\alpha] = \\ &= \begin{bmatrix} {}^{\mathrm{ll}}\mathbf{D} + {}^{\mathrm{pp}}\mathbf{D} & {}^{\mathrm{pi}}\mathbf{D} \\ {}^{\mathrm{ip}}\mathbf{D} & {}^{\mathrm{ii}}\mathbf{D} \end{bmatrix} + \left(\begin{bmatrix} {}^{\mathrm{pl}}\mathbf{D} & \mathbf{0} \\ {}^{\mathrm{il}}\mathbf{D} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {}^{\mathrm{lp}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \cos{}_\beta\alpha\ , \end{aligned} \tag{23}$$

$${}_\beta\mathbf{H}_2 = (\mathbf{G} - \mathbf{E})\sin{}_\beta\alpha = \left(\begin{bmatrix} {}^{\mathrm{lp}}\mathbf{D} & {}^{\mathrm{li}}\mathbf{D} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} {}^{\mathrm{pl}}\mathbf{D} & \mathbf{0} \\ {}^{\mathrm{il}}\mathbf{D} & \mathbf{0} \end{bmatrix} \right) \sin{}_\beta\alpha\ . \tag{24}$$

The characteristic vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ can be determined from the relation (22), natural frequencies of the whole periodic system can be determined from the condition of a nontrivial solution:

$$\det{}_\beta\mathbf{H} = \det \begin{bmatrix} {}_\beta\mathbf{H}_1 & {}_\beta\mathbf{H}_2 \\ -{}_\beta\mathbf{H}_2 & {}_\beta\mathbf{H}_1 \end{bmatrix} = 0\ . \tag{25}$$

Mode shapes of the whole periodic system can be calculated by substituting the characteristic vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ into equation (15). Constants C_1 and C_2 when enumerating mode shapes ${}^{\mathrm{pi}}\mathbf{U}_k$ can be selected arbitrarily. It follows from both the decision procedure itself and the definition of mode shapes, which must form a linearly independent basis. During the visualization of the particular mode shapes of the real steam turbine bladed disk [1], [8] conditions $C_1 = 1$ and $C_2 = 0$ are applied similarly as in [14].

Generally natural vectors ${}_\beta\mathbf{a}$ and ${}_\beta\mathbf{b}$ are different for identical β. It can be shown that with certain types of the subsystem symmetry it holds ${}_\beta\mathbf{a} = c\,{}_\beta\mathbf{b}$ (where $\beta = 1, 2, \ldots, M-1$; when M is even with the condition $\beta \neq M/2$) and relations for the calculations of natural frequencies and natural mode shapes of the rotationally periodic system are simpler [14].

When formulating the initial difference equations for the calculations of the subvectors of displacements amplitudes ${}^{\mathrm{p}}\mathbf{U}_k$ and ${}^{\mathrm{i}}\mathbf{U}_k$ and the subvector of amplitudes of the generalized coupling forces ${}^{\mathrm{p}}\mathbf{Q}_k$ (relations (7), (8) and (9)) for cyclic systems it is necessary to respect the influence of the subsystem geometry, of the geometry of the whole periodic system and of the option of the local coordinate system, in which the coordinates of the subsystem points are determined, on the specific formulations of the compatibility conditions (4) and the conditions of equilibrium of the generalized coupling forces (5).

In the rotationally periodic system the subsystem of a sector shape will be considered (see Fig. 2). The sector angle is designated 2γ. The system of coordinates, in which the coordinates of points, the components of the vector of displacements amplitudes $\mathbf{U}_k$ and the components of the vector of the generalized coupling forces $\mathbf{Q}_k$ amplitudes are determined, will be a local right-handed Cartesian coordinate system xyz in the k-th subsystem. The coordinate system will be located in such a way so that z axis may be the axis of the symmetry of the points in which the k-th subsystem is coupled with the subsystem $k-1$ (adjacent to the left) and with the subsystem $k+1$ (adjacent to the right) and x axis may be the axis of the rotational symmetry of the whole rotationally periodic system. Further, auxiliary local right-handed Cartesian coordinate systems ${}^{\mathrm{l}}x\,{}^{\mathrm{l}}y\,{}^{\mathrm{l}}z$ and ${}^{\mathrm{p}}x\,{}^{\mathrm{p}}y\,{}^{\mathrm{p}}z$ (see Fig. 2) will be introduced in the k-th subsystem.

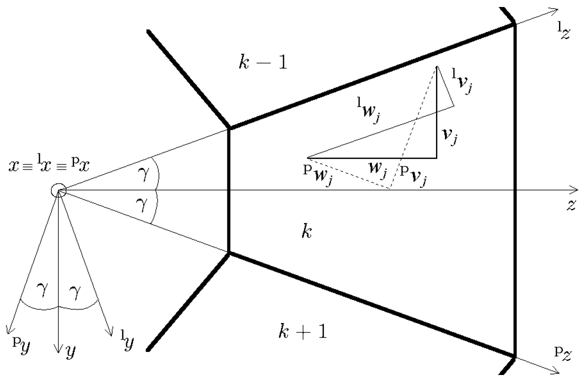

Fig.2: Scheme of the sector shaped subsystem in the rotationally periodic system

In the same way as in the k-th subsystem local right-handed Cartesian coordinate systems xyz, ${}^{\mathrm{l}}x\,{}^{\mathrm{l}}y\,{}^{\mathrm{l}}z$ and ${}^{\mathrm{p}}x\,{}^{\mathrm{p}}y\,{}^{\mathrm{p}}z$ will also be introduced in the subsystems $k-1$ and $k+1$. In the points common to the subsystems $k-1$ and k the coordinate systems of ${}^{\mathrm{p}}x\,{}^{\mathrm{p}}y\,{}^{\mathrm{p}}z$ of the $k-1$ subsystem and ${}^{\mathrm{l}}x\,{}^{\mathrm{l}}y\,{}^{\mathrm{l}}z$ of the k-th subsystem are identical. In the points common to the subsystems k and $k+1$ the coordinate systems of ${}^{\mathrm{p}}x\,{}^{\mathrm{p}}y\,{}^{\mathrm{p}}z$ of the k-th subsystem and ${}^{\mathrm{l}}x\,{}^{\mathrm{l}}y\,{}^{\mathrm{l}}z$ of the $k+1$ subsystem are identical. Then in the rotationally periodic system the compatibility conditions (4) are of the form

$$\begin{aligned} \mathbf{T}(-\gamma)\,{}^{\mathrm{p}}\mathbf{U}_{k-1} &= \mathbf{T}(\gamma)\,{}^{\mathrm{l}}\mathbf{U}_k\ , \\ \mathbf{T}(-\gamma)\,{}^{\mathrm{p}}\mathbf{U}_k &= \mathbf{T}(\gamma)\,{}^{\mathrm{l}}\mathbf{U}_{k+1} \end{aligned} \tag{26}$$

and the conditions of equilibrium of the generalized coupling forces (5) :

$$\begin{aligned} \mathbf{T}(-\gamma)\,{}^{\mathrm{p}}\mathbf{Q}_{k-1} &= -\mathbf{T}(\gamma)\,{}^{\mathrm{l}}\mathbf{Q}_k\ , \\ \mathbf{T}(-\gamma)\,{}^{\mathrm{p}}\mathbf{Q}_k &= -\mathbf{T}(\gamma)\,{}^{\mathrm{l}}\mathbf{Q}_{k+1}\ . \end{aligned} \tag{27}$$

where $\mathbf{T}(-\gamma)$ and $\mathbf{T}(\gamma)$ are the square transformation matrices.

Matrix $\mathbf{T}(\gamma)$ is of the form

$$\mathbf{T}(\gamma) = \begin{bmatrix} \mathbf{t}_1(\gamma) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_2(\gamma) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{t}_N(\gamma) \end{bmatrix} \tag{28}$$

where submatrix $\mathbf{t}_j(\gamma)$, $j = 1, 2, \ldots, N$, in the most general case of six degrees of freedom in the point, is of the shape

$$\mathbf{t}_j(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma & 0 & 0 & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix} . \tag{29}$$

Transformation matrix $\mathbf{T}(-\gamma)$ is similar to the matrix $\mathbf{T}(\gamma)$, it contains the angle $-\gamma$ instead of the angle γ. Between the matrices $\mathbf{T}(\gamma)$ and $\mathbf{T}(-\gamma)$ there is relation

$$\mathbf{T}(\gamma)\,\mathbf{T}(-\gamma) = \mathbf{I} . \tag{30}$$

When determining the subsystem geometry in the shape of a sector in the considered local right-handed Cartesian coordinate systems xyz the compatibility conditions (4) and the conditions of equilibrium of the generalized coupling forces (5) are replaced with relations (26) and (27).

When assembling the equations for the calculation of the natural vibration characteristics of the linear undamped rotationally periodic system according to the described method the solution of the matrix equation (17) or (22) leads to the generalized eigenvalue problem [1]. The equation of the following type will be solved (in accordance with equation (13); $\beta = 0, 1, \ldots, M-1$) :

$$_\beta\mathbf{H}\,_\beta\mathbf{v} = (_\beta\mathbf{H}_\mathrm{K} - {}_\beta\omega^2\,_\beta\mathbf{H}_\mathrm{M})\,_\beta\mathbf{v} = \mathbf{0} . \tag{31}$$

The subspace iteration method (e.g. [25]), which enables to determine the selected number of the lowest natural frequencies q and corresponding natural vectors, is a very efficient numerical method. This method requires the matrices $_\beta\mathbf{H}_\mathrm{K}$ and $_\beta\mathbf{H}_\mathrm{M}$ to be symmetric and in addition matrix $_\beta\mathbf{H}_\mathrm{K}$ to be positive definite (at the same time those conditions guarantee the eigenvalues $_\beta\omega^2$ of equation (31) to be real positive). The proof that matrices $_\beta\mathbf{H}_\mathrm{K}$ and $_\beta\mathbf{H}_\mathrm{M}$ comply with the stated conditions is carried out in [1] ($\beta = 0, 1, \ldots, M-1$).

After the determination of the values $_\beta\omega_j^2$ ($j = 1, 2, \ldots, q$) the natural frequencies (in [Hz]) of the whole periodic system can be calculated from the relation

$$_\beta f_j = \frac{\sqrt{_\beta\omega_j^2}}{2\pi} , \qquad j = 1, 2, \ldots, q . \tag{32}$$

5. Conclusions

The PERCOK in-house software [8] was created in the Compaq Visual Fortran programming language on the basis of the relations for the calculation of the natural vibration characteristics of the linear undamped rotationally periodic systems with the subsystems of a sector shape. The PERCOK software uses the subsystem stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$ assembled applying the COSMOS/M FEM software. The PERG in-house software [8], in the Compaq Visual Fortran programming language as well, was created to visualize the natural mode shapes of the rotationally periodic systems. Those computer softwares were originally [1] created in the Fortran 5.0 language, which is applicable only under the MS-DOS operating system. That is why the operating memory, which could be utilized by the softwares, was limited and it was possible to operate only with the matrices up to the certain order (the subsystem model could have 510 degrees of freedom at maximum). By reprogramming to the Compaq Visual Fortran language, which works under the Windows operating system, the matrices order and thus the subsystem model discretization are only limited by the computer operating memory capacity. Those in-house softwares were applied in the calculation of the natural vibration characteristics of the steam turbine bladed disk with the blades of the ZN340-2 type with the continuous binding by shrouding and a tie-boss [8] (the subsystem model has 9 537 degrees of freedom).

The respecting of the material damping properties or gyroscopic effects are the possible ways of further development of the given problematics. First a qualified verification of the extent of the influence of their consideration on the natural vibration characteristics of the rotationally periodic systems (specifically the bladed disks) will have to be performed. Due to the material damping properties or gyroscopic effects considering not only the way of the solution of the matrix difference equations (7) to (9) is changed but at the same time using the commercial software [26] is not possible for assembling the damping or gyroscopic matrices. Further possibility is the consideration of 'disturbing' the system periodicity because of the structural deviations of some subsystems (in this case it does not hold any more that the order of stiffness and mass matrices of the whole system is double at most in comparison with the order of stiffness and mass matrices of the single subsystem) or the solution of the forced vibration.

Acknowledgement

The article has originated in the framework of the solution of the Research Plan of the Ministry of Education, Youth and Sports of the Czech Republic MSM4771868401.

References

[1] Polach P.: Výpočet charakteristik vlastního kmitání olopatkovaných disků parních turbin s využitím rotační periodicity (Calculating Eigenfrequencies and Mode Shapes of Steam Turbines' Bladed Disks by Utilization of the Rotational Periodicity), PhD Thesis, FAS UWB in Pilsen 1998, Plzeň (in Czech)

[2] Polach P.: Výpočet vlastního kmitání disků a olopatkovaných disků parních turbin pomocí subsystémů (Calculation of Free Vibration of the Disks and Bladed Disks by Means of Subsystems), In: Colloquium Computational Mechanics-93, Department of Mechanics UWB in Pilsen, Pernink 1993, p. 155 (in Czech)

[3] Polach P.: Calculation of Free Vibration of Steam Turbine's Acting Wheel by Using the Rotational Periodicity and Comparison with Measurement, In: National Conference with International Participation Engineering Mechanics '99, eds. C. Kratochvíl, V. Kotek, J. Krejsa, ISM FME TU of Brno, Svratka 1999, Vol. 1, p. 199

[4] Polach P.: Analysis of Dynamic Properties of Steam Turbine's Acting Wheel with Blades Connected by a Continuous Binding, In: VIII. International Conference on the Theory of Machines and Mechanisms, eds. J. Mrázek, J. Kaniok, M. Bílek, TU of Liberec, Liberec 2000, p. 631

[5] Chmelíček M.: Frekvenčně modální vlastnosti lopatky 220Z-1085 s cik-cak vazbou a volné (Frequency Modal Properties of the 220Z-1085 Blade with the Zigzag Binding and Free One), Research Report ŠKODA, VÝZKUM, s.r.o., VZVÚ 0779, 1994, Plzeň (in Czech)

[6] Polach P.: Výpočet charakteristik vlastního kmitání lineárních netlumených rotačně periodických systémů (Calculation of the Characteristics of Natural Vibration of Linear Undamped Rotationally Periodic Systems), In: National Colloquium with International Participation Dynamics of Machines 2007, ed. L. Pešek, Institute of Thermomechanics AS CR, Praha 2007, p. 143 (in Czech)

[7] Polach P.: Calculation of natural vibration of linear undamped rotationally periodic structures considering elimination of degrees of freedom, In: National Colloquium with International Participation Dynamics of Machines 2008, ed. L. Pešek, Institute of Thermomechanics AS CR, Prague 2008, p. 135
[8] Polach P., Jankovec J., Kindelmann P.: Výpočet dynamických charakteristik oběžného kola parní turbíny s oběžnými lopatkami s průběžnou vazbou bandáží a tie-bossem (Calculation of Dynamic Characteristics of the Steam Turbine Bladed Disk with the Blades with the Continuous Binding by Shrouding and Tie-boss), Research Report ŠKODA VÝZKUM s.r.o., VYZ 0904/06, 2006, Plzeň (in Czech)
[9] Mester S.S., Benaroya H.: Periodic and Near-Periodic Structures, J. Shock and Vibration 2 (1995) 69
[10] Mead D.J.: Wave Propagation in Continuous Periodic Structures: Research Contributions from Southampton, 1964–1995, J. of Sound and Vibration, 190 (1996) 495
[11] Feather N.: An Introduction to the Physics of Vibrations and Waves, Edinburgh University Press 1961, Edinburgh
[12] Pětrovský V.: Vlastní kmitání periodických mechanických systémů, 1. Odvození základních vztahů a řešení uzavřených systémů (Natural Vibration of the Periodic Mechanical Systems, 1. Derivation of Basic Relations and Solution of Closed Systems), Research Report ÚTSSK Plzeň, 55VP, 1988, Plzeň (in Czech)
[13] Pětrovský V.: Vlastní kmitání periodických mechanických systémů, 2. Konečné spojité systémy a systémy diskrétní (Natural Vibration of the Periodic Mechanical Systems, 2. Finite Continuous Systems and Discrete Systems), Research Report ÚTSSK Plzeň, 80VP, 1990, Plzeň (in Czech)
[14] Pětrovský V.: Vlastní kmitání symetrických periodických systémů (Natural Vibration of Symmetric Periodic Systems), Research Report ŠKODA s.p., Plzeň, ÚVZÚ, VZVÚ 0473, 1991, Plzeň (in Czech)
[15] Kempner M.L., Efremova V.T.: Kolebanija bandažirovannych lopatok turbomašin (Vibrations of Shrouded Blades of Turbines), separate, pp. 42–52 (in Russian)
[16] Fricker A.J., Potter S.: Transient Forced Vibration of Rotationally Periodic Structures, Int. J. for Numerical Method in Engineering, 17 (1981) 957
[17] Arduini C.: An Approach to the Dynamics of Modular Repetitive Structures, Acta Astronautica, 12 (1985) 661
[18] DiMaggio S.J., Duron Z.H., Davis, G.: Modal Identification of Lightly-Damped, Highly-Symmetric Bladed Disks, In: AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, AIAA, Denver 2002, CD-ROM
[19] Dubigeon S., Michon J.C.: Modes for Deformable Periodic Cyclic Symmetric Systems Driven in Uniform Rotation by a Flexible Shaft, J. of Sound and Vibration, 106 (1986) 53
[20] Krásný I.: Numerical Solution of Rotationally Periodic Structures, Acta Technica ČSAV, 27 (1982) 634
[21] Krásný I.: Redukce neznámých při výpočtech rotačně periodických konstrukcí (Reduction of Unknowns in the Calculations of Rotationally Periodic Structures), In: Conference on Mathematical Models in Engineering Science, Karlovy Vary 1986, p. I/267 (in Czech)
[22] Krásný I., Mančal J.: Dynamics of Bladed Disc of General Shape, In: Sixteenth International Conference on Dynamics of Machines, Stupava 1989, p. 254
[23] Burda Z.: Vibration of Rotationally Periodic Structures, In: Sixteenth International Conference on Dynamics of Machines, Stupava 1989, p. 254
[24] Prágerová A.: Diferenční rovnice (Difference Equations), SNTL Publishing House 1971, Praha (in Czech)
[25] Bittnar Z., Řeřicha P.: Metoda konečných prvků v dynamice konstrukcí (Finite Element Method in Dynamics of Structures), SNTL Publishing House 1981, Praha (in Czech)
[26] Kellner J.: Modální vlastnosti svazků turbínových lopatek (Modal Properties of Packets of Turbine Blades), Essay to State PhD Examination, Department of Mechanics FAS UWB in Pilsen 2006, Plzeň (in Czech)

Appendix A – Solution of matrix difference equations

The matrices **E**, **F** and **G** (see equation (11) in the main article) consist of the submatrices of the dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem and zero submatrices. As the dynamic stiffness matrix $\mathbf{D}(\omega)$ for all the subsystems is identical the matrices **E**, **F** and **G** are not dependent on k. The matrix equation (11) thus constitutes the system of homogeneous linear difference equations of the second type of the second order with constant coefficients [24]. That is why (according to the theory of difference equations [24]) the particular solutions of the matrix difference equation (11) are of the form

$$^{\mathrm{pi}}\mathbf{U}_k = \mathbf{v}\,\lambda^k \tag{A1}$$

where vector $\mathbf{v}$ is of the same dimension as vector of displacements amplitudes $^{\mathrm{pi}}\mathbf{U}_k$ and is independent of k; λ ($\neq 0$) is the unknown constant for the moment (λ^k is the k-th power of λ).

Substitution of the assumed form of the solution (A1) into the equation (11) gives the equation

$$(\mathbf{E}\,\lambda^{k-1} + \mathbf{F}\,\lambda^k + \mathbf{G}\,\lambda^{k+1})\,\mathbf{v} = \mathbf{0}\ . \tag{A2}$$

As the nontrivial solution $\lambda \neq 0$ is searched for, equation (A2) can be divided by the number λ^{k-1} :

$$(\mathbf{E} + \mathbf{F}\,\lambda + \mathbf{G}\,\lambda^2)\,\mathbf{V} = \mathbf{0}\ . \tag{A3}$$

Since solution for $\mathbf{v} \neq \mathbf{0}$ is searched for, following condition for the calculation of constant λ is valid:

$$\det(\mathbf{E} + \mathbf{F}\,\lambda + \mathbf{G}\,\lambda^2) = 0\ . \tag{A4}$$

This equation is the algebraic (characteristic) equation of the $2n$-th degree (n is dimension of the vector $^{\mathrm{pi}}\mathbf{U}_k$, or order of the matrices **E**, **F** and **G**), which has $2\,n$ non zero roots $_j\lambda$, where $j = 1, 2, \ldots, 2\,n$. Roots of this equation can be both real and complex numbers.

It is assumed that m ($m \leq 2\,n$) roots λ of the characteristic equation (A4) are complex. Complex root $_L\lambda$ can be expressed in the goniometric form

$$_L\lambda = {_L r}\,(\cos {_L\alpha} + \mathrm{i}\,\sin {_L\alpha})\ . \tag{A5}$$

Due to further process of solution an assumption that the matrices **E**, **F** and **G** have only real elements is introduced.

The matrices **E**, **F** and **G** contain the elements of the dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem. In order that the matrices **E**, **F** and **G** may have only real elements the dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem must also have only real elements. The dynamic stiffness matrix $\mathbf{D}(\omega)$ of the subsystem must not contain the damping matrix **B** of the subsystem, i.e. further an undamped periodic system will be considered (if a damped periodic system were considered the term $\mathrm{i}\,\omega\,\mathbf{B}$ containing the imaginary unit i would be added in the equation (A6)):

$$\mathbf{D}(\omega) = \mathbf{K} - \omega^2\,\mathbf{M} \tag{A6}$$

where **K** is the stiffness matrix of the subsystem and **M** is the mass matrix of the subsystem.

In case that the matrices **E**, **F** and **G** contain only real elements even equation (A4) has only real coefficients and it holds: if equation (A4) has complex root ${}_L\lambda$, it also has complex conjugate root [24]

$$ {}_{L+1}\lambda = {}_L r\,(\cos {}_L\alpha - \mathrm{i}\,\sin {}_L\alpha) \tag{A7} $$

and a number of complex roots of the characteristic equation (A4) m must be an even number.

If the characteristic vector

$$ {}_L\mathbf{v} = {}_L\mathbf{a} + \mathrm{i}\,(-{}_L\mathbf{b}) \tag{A8} $$

corresponds to the characteristic value ${}_L\lambda$ then the complex conjugate characteristic vector

$$ {}_{L+1}\mathbf{v} = {}_L\mathbf{a} + \mathrm{i}\,{}_L\mathbf{b} \tag{A9} $$

corresponds to the characteristic value ${}_{L+1}\lambda$.

By substituting relations (A5), (A7), (A8) and (A9) into equation (A1) the particular solution of equation (11) for the characteristic values ${}_L\lambda$ and ${}_{L+1}\lambda$ is obtained:

$$ {}^{\mathrm{pi}}_{L}\mathbf{U}_k = {}_L r^k\,[\cos({}_L\alpha\,k) + \mathrm{i}\,\sin({}_L\alpha\,k)][{}_L\mathbf{a} + \mathrm{i}\,(-{}_L\mathbf{b})]\ , \tag{A10} $$

$$ {}^{\mathrm{pi}}_{L+1}\mathbf{U}_k = {}_L r^k\,[\cos({}_L\alpha\,k) - \mathrm{i}\,\sin({}_L\alpha\,k)]({}_L\mathbf{a} + \mathrm{i}\,{}_L\mathbf{b})\ . \tag{A11} $$

On the basis of the theorems on the multiplication of the particular solution by the constant and on the sum of the particular solution of homogeneous linear difference equations of the second type [24] it is possible to prove that the functions

$$ \frac{1}{2}\left({}^{\mathrm{pi}}_{L}\mathbf{U}_k + {}^{\mathrm{pi}}_{L+1}\mathbf{U}_k\right) = {}_L r^k\left[\cos({}_L\alpha\,k)\,{}_L\mathbf{a} + \mathrm{i}\,\sin({}_L\alpha\,k)\,{}_L\mathbf{b}\right]\ , \tag{A12} $$

$$ \frac{1}{2\,\mathrm{i}}\left({}^{\mathrm{pi}}_{L}\mathbf{U}_k - {}^{\mathrm{pi}}_{L+1}\mathbf{U}_k\right) = {}_L r^k\left[\cos({}_L\alpha\,k)\,(-{}_L\mathbf{b}) + \mathrm{i}\,\sin({}_L\alpha\,k)\,{}_L\mathbf{a}\right]\ , \tag{A13} $$

also are the solution of equation (11).

Relation (A12) corresponds to the real parts of the complex particular solutions ${}^{\mathrm{pi}}_{L}\mathbf{U}_k$ and ${}^{\mathrm{pi}}_{L+1}\mathbf{U}_k$, relation (A13) corresponds to the imaginary parts of the complex particular solution ${}^{\mathrm{pi}}_{L}\mathbf{U}_k$ and ${}^{\mathrm{pi}}_{L+1}\mathbf{U}_k$. A linear independency of the homogeneous linear difference equations of the second type is the requirement for their particular solution, from the point of view of their substituting into the general solution. The condition of the linear independency is also fulfilled by relations (A12) and (A13). Relations (A12) and (A13) instead of relations (A10) and (A11) can be taken for the particular solutions of equation (11) when not complex but real functions are the solution.

After the formal redesignation (instead of $({}^{\mathrm{pi}}_{j}\mathbf{U}_k + {}^{\mathrm{pi}}_{j+1}\mathbf{U}_k)/2$ it will be written ${}^{\mathrm{pi}}_{j}\mathbf{U}_k$ and instead of $({}^{\mathrm{pi}}_{j}\mathbf{U}_k + {}^{\mathrm{pi}}_{j+1}\mathbf{U}_k)/(2\,\mathrm{i})$ it will be written ${}^{\mathrm{pi}}_{j+1}\mathbf{U}_k$) altogether $m/2$ pairs of particular solutions of equation (11) will be obtained:

$$ \begin{aligned} {}^{\mathrm{pi}}_{j}\mathbf{U}_k &= {}_j r^k\left[\cos({}_j\alpha\,k)\,{}_j\mathbf{a} + \mathrm{i}\,\sin({}_j\alpha\,k)\,{}_j\mathbf{b}\right]\ , \\ {}^{\mathrm{pi}}_{j+1}\mathbf{U}_k &= {}_L r^k\left[\cos({}_j\alpha\,k)\,(-{}_j\mathbf{b}) + \mathrm{i}\,\sin({}_j\alpha\,k)\,{}_j\mathbf{a}\right] \quad \text{where} \quad j = 1, 3, 5, \ldots, m-1\ . \end{aligned} \tag{A14} $$

In agreement with the already mentioned assumption the characteristic equation (A4) has $2\,n - m$ real roots ${}_j\lambda$, where $j = m+1, m+2, \ldots, 2\,n$. The real roots of equation (A4) can be obtained from the expression in the goniometric form (A5). It is obvious that for the real roots it must be ${}_j\alpha = l\,\pi$, $l = 0,1,2,\dots$; $j = m+1, m+2, \dots, 2\,n$. The root expressed in the form (A5) corresponds with the characteristic vector in the form (A8). For real roots ${}_j\lambda$ ($j = m+1, m+2, \dots, 2\,n$) the imaginary part of this characteristic vector $(-{}_j\mathbf{b})$ ($j = m+1, m+2, \dots, 2\,n$) has no meaning. Then the particular solution of equation (11) can be written for real roots in the form (according to relation (A10)):

$$ {}^{\rm pi}_{j}\mathbf{U}_k = {}_j r^k \cos({}_j\alpha\,k)\,{}_j\mathbf{a} \quad \text{where} \quad j = m+1, m+2, \dots, 2\,n \; . \tag{A15} $$

It is possible that some complex root of the characteristic equation (A4) is s-fold. It is assumed that this root is complex root ${}_1\lambda = {}_1r\,(\cos{}_1\alpha + \mathrm{i}\,\sin{}_1\alpha)$. Particular solution of equation (11) ${}^{\ \ \ \ \ \ \rm pi}_{1,2,\dots,s}\mathbf{U}_k$ for s-fold root ${}_1\lambda$ will be of the form

$$ {}^{\ \ \ \ \ \ \rm pi}_{1,2,\dots,s}\mathbf{U}_k = {}_j\lambda^k\,{}_1\mathbf{p} \tag{A16} $$

where ${}_1\mathbf{p} = {}_1\mathbf{c}_0 + k\,{}_1\mathbf{c}_1 + \dots + k^{s-1}\,{}_1\mathbf{c}_{s-1}$ is the vector polynomial of $s-1$ degree in variable k corresponding with the s-fold characteristic value ${}_1\lambda$. The vector polynomial ${}_1\mathbf{p}$ can be put down in the form

$$ {}_1\mathbf{p} = {}_1\mathbf{p}_{\rm Re} + \mathrm{i}\,(-{}_1\mathbf{p}_{\rm Im}) \tag{A17} $$

where ${}_1\mathbf{p}_{\rm Re} = {}_1\mathbf{a}_0 + k\,{}_1\mathbf{a}_1 + \dots + k^{s-1}\,{}_1\mathbf{a}_{s-1}$ and ${}_1\mathbf{p}_{\rm Im} = {}_1\mathbf{b}_0 + k\,{}_1\mathbf{b}_1 + \dots + k^{s-1}\,{}_1\mathbf{b}_{s-1}$.

If the complex root ${}_1\lambda$ is s-fold root of the characteristic equation (A4) then complex conjugate root ${}_2\lambda = {}_1r(\cos{}_1\alpha - \mathrm{i}\,\sin{}_1\alpha)$ must also be s-fold. Vector polynomial ${}_2\mathbf{p}$ corresponding with the characteristic value ${}_2\lambda$ is also of $s-1$ degree in variable k and is complex conjugate to vector polynomial ${}_1\mathbf{p}$:

$$ {}_2\mathbf{p} = {}_1\mathbf{p}_{\rm Re} - \mathrm{i}\,(-{}_1\mathbf{p}_{\rm Im}) = {}_1\mathbf{p}_{\rm Re} + \mathrm{i}\,{}_1\mathbf{p}_{\rm Im} \; . \tag{A18} $$

When substituting relation (A17) into equation (A16) and relation (A18) into the similar equation for root ${}_2\lambda$ then particular solutions of equation (11) for s-fold complex conjugate characteristic values ${}_1\lambda$ and ${}_2\lambda$ are obtained:

$$ {}^{\ \ \ \ \ \ \rm pi}_{1,2,\dots,s}\mathbf{U}_k = {}_1r^k\,[\cos({}_1\alpha\,k) + \mathrm{i}\,\sin({}_1\alpha\,k)]\,[{}_1\mathbf{p}_{\rm Re} + \mathrm{i}\,(-{}_1\mathbf{p}_{\rm Im})] \; , \tag{A19} $$

$$ {}^{\ \ \ \ \ \ \ \ \ \ \ \ \ \rm pi}_{s+1,s+2,\dots,2s}\mathbf{U}_k = {}_1r^k\,[\cos({}_1\alpha\,k) - \mathrm{i}\,\sin({}_1\alpha\,k)]\,({}_1\mathbf{p}_{\rm Re} + \mathrm{i}\,{}_1\mathbf{p}_{\rm Im}) \; . \tag{A20} $$

Similarly as relations (A10) and (A11) for the simple complex conjugate roots, equations (A19) and (A20) can be written into the relations corresponding with the real parts of complex particular solutions ${}^{\ \ \ \ \ \ \rm pi}_{1,2,\dots,s}\mathbf{U}_k$ and ${}^{\ \ \ \ \ \ \ \ \ \ \ \ \ \rm pi}_{s+1,s+2,\dots,2s}\mathbf{U}_k$ and the imaginary parts of the complex particular solutions ${}^{\ \ \ \ \ \ \rm pi}_{1,2,\dots,s}\mathbf{U}_k$ and ${}^{\ \ \ \ \ \ \ \ \ \ \ \ \ \rm pi}_{s+1,s+2,\dots,2s}\mathbf{U}_k$, which further will be considered to be the particular solutions of equations (A4):

$$ \begin{aligned} {}^{\ \ \ \ \ \ \ \ \ \rm pi}_{1,3,\dots,2s-1}\mathbf{U}_k &= {}_1r^k\,[\cos({}_1\alpha\,k)\,{}_1\mathbf{p}_{\rm Re} + \sin({}_1\alpha\,k)\,{}_1\mathbf{p}_{\rm Im}] \; , \\ {}^{\ \ \ \ \ \ \rm pi}_{2,4,\dots,2s}\mathbf{U}_k &= {}_1r^k\,[\cos({}_1\alpha\,k)\,(-{}_1\mathbf{p}_{\rm Im}) + \sin({}_1\alpha\,k)\,{}_1\mathbf{p}_{\rm Re}] \; . \end{aligned} \tag{A21} $$

Further it will be considered that real root ${}_{m+1}\lambda$ of characteristic equation (A4) is t-fold. When comparing and modifying relation (A19) for the real roots it is obvious that

${}_{m+1}\alpha = l\,\pi,\ l = 0,1,2,\ldots$ and vector polynomial ${}_{m+1}\mathbf{p}_{\mathrm{Im}}$ has no meaning for real roots. Polynomial ${}_{m+1}\mathbf{p} = {}_{m+1}\mathbf{a}_0 + k\,{}_{m+1}\mathbf{a}_1 + \cdots + k^{t-1}\,{}_{m+1}\mathbf{a}_{t-1}$ is the vector polynomial of $t-1$ degree in variable k corresponding with t-fold characteristic value ${}_{m+1}\lambda$. The particular solution of equation (11) ${}_{m+1,m+2,\ldots,m+t}^{\mathrm{pi}}\mathbf{U}_k$ for multiple real root ${}_{m+1}\lambda$ will be of the form

$$ {}_{m+1,m+2,\ldots,m+t}^{\mathrm{pi}}\mathbf{U}_k = {}_{m+1}r^k\ \cos({}_{m+1}\alpha\,k)\,{}_{m+1}\mathbf{p}\ . \tag{A22}$$

The general solution of equation (11) is obtained as linear combination of all the particular solutions given by relations (A14), (A15), (A21) and (A22). For the reason of the simpler notation it will be considered that in equation (11) only complex conjugate roots ${}_1\lambda$ and ${}_2\lambda$ and real root ${}_{m+1}\lambda$ are multiple (s-fold and t-fold):

$$\begin{aligned}
{}^{\mathrm{pi}}\mathbf{U}_k = {}_1r^k\Big\{&\cos({}_1\alpha\,k)\,(C_1\,{}_1\mathbf{a}_0 + C_3\,k\,{}_1\mathbf{a}_1 + \cdots + C_{2s-1}\,k^{s-1}\,{}_1\mathbf{a}_{s-1}) + \\
&+ \sin({}_1\alpha\,k)\,(C_1\,{}_1\mathbf{b}_0 + C_3\,k\,{}_1\mathbf{b}_1 + \cdots + C_{2s-1}\,k^{s-1}\,{}_1\mathbf{b}_{s-1}) + \\
&+ \cos({}_1\alpha\,k)\,[C_2\,(-{}_1\mathbf{b}_0) + C_4\,k\,(-{}_1\mathbf{b}_1) + \cdots + C_{2s}\,k^{s-1}\,(-{}_1\mathbf{b}_{s-1})] + \\
&+ \sin({}_1\alpha\,k)\,(C_2\,{}_1\mathbf{a}_0 + C_4\,k\,{}_1\mathbf{a}_1 + \cdots + C_{2s}\,k^{s-1}\,{}_1\mathbf{a}_{s-1})\Big\} + \\
&+ \sum_{j=s+1}^{m/2} {}_{2j-1}r^k\,C_{2j-1}\,[\cos({}_{2j-1}\alpha\,k)\,{}_{2j-1}\mathbf{a} + \sin({}_{2j-1}\alpha\,k)\,{}_{2j-1}\mathbf{b}] + \\
&+ \sum_{j=s+1}^{m/2} {}_{2j-1}r^k\,C_{2j}\,[\cos({}_{2j-1}\alpha\,k)\,(-{}_{2j-1}\mathbf{b}) + \sin({}_{2j-1}\alpha\,k)\,{}_{2j-1}\mathbf{a}] + \\
&+ {}_{m+1}r^k\ \cos({}_{m+1}\alpha\,k)\,\big(C_{m+1}\,{}_{m+1}\mathbf{a}_0 + C_{m+2}\,k\,{}_{m+1}\mathbf{a}_1 + \\
&\qquad\qquad + \cdots + C_{m+t}\,k^{t-1}\,{}_{m+1}\mathbf{a}_{t-1}\big) + \\
&+ \sum_{j=m+t+1}^{2n} {}_jr^k\ \cos({}_j\alpha\,k)\,C_j\,{}_j\mathbf{a}\ .
\end{aligned} \tag{A23}$$

The general solution must fulfil the cyclic condition, which must be fulfilled for each of the total number of M subsystems:

$$ {}^{\mathrm{pi}}\mathbf{U}_{k+M} = {}^{\mathrm{pi}}\mathbf{U}_k \quad \text{where} \quad k = 1,2,\ldots,M\ . \tag{A24}$$

By substituting relation (A23) for the k-th subsystem and relation (A23) rewritten for the $k+M$-th subsystem into the cyclic condition (A24) and by their mutual comparing it can be found out:

1. the searched solution does not contain multiple roots (constants C_3, $C_4, \ldots,$ C_{2s-1}, C_{2s}, $C_{m+2}, \ldots,$ C_{m+t} are zero),
2. ${}_jr = 1$, independently of j ($j = 1,3,5,\ldots,m-1,m+1,m+2,\ldots,2\,n$),
3. $\cos[{}_j\alpha\,(k+M)] = \cos({}_j\alpha\,k)$, where $j = 1,3,5,\ldots,m-1,m+1,m+2,\ldots,2\,n$ and $\sin[{}_j\alpha\,(k+M)] = \sin({}_j\alpha\,k)$, where $j = 1,3,5,\ldots,m-1$.

In order that both conditions in point 3 may be valid at the same time there must be (independently of j): ${}_j\alpha = 2\pi\,\beta/M$, where $\beta = 0,1,\ldots,M-1$. The value of angle ${}_j\alpha$ is dependent on the optional parameter β (for the mode shapes of the rotationally periodic systems the parameter β means the number of nodal diameters). For the solution fulfilling the cyclic condition it is not necessary to write index j either at angle ${}_j\alpha$ or at the module of the complex number ${}_jr$:

$$r = {}_jr = 1 \qquad \text{and} \qquad {}_\beta\alpha = {}_j\alpha = \frac{2\pi\,\beta}{M} \ , \quad \text{where} \quad \beta = 0, 1, \dots, M-1 \ . \tag{A25}$$

Two forms of equations are obtained for the calculation of natural frequencies and mode shapes of the whole periodic system in dependence on the parameter β value: the first form of equations (see relations from (14) to (19) in the main article) holds for $\beta = 0$ and in addition in case of even M $\beta = M/2$, the second form of equations (see relations from (20) to (25) in the main article) holds for $\beta = 0, 1, \dots, M-1$ and when M is even with the condition $\beta \neq M/2$.

Appendix B – Angle difference identities

Expression $\cos[{}_\beta\alpha\,(k-1)]$ will be substituted by the expression obtained according to the well known angle difference identities in equation (16) (${}_\beta\alpha = 2\,\pi\,\beta/M$, where $\beta = 0$ and in addition in case of even M $\beta = M/2$) :

$$\cos[{}_\beta\alpha\,(k-1)] = \cos({}_\beta\alpha\,k)\,\cos{}_\beta\alpha + \sin({}_\beta\alpha\,k)\,\sin{}_\beta\alpha = \cos({}_\beta\alpha\,k)\cos{}_\beta\alpha \tag{B1}$$

and $\cos[{}_\beta\alpha\,(k+1)]$ by the expression obtained according to the well known angle sum identities :

$$\cos[{}_\beta\alpha\,(k+1)] = \cos({}_\beta\alpha\,k)\,\cos{}_\beta\alpha - \sin({}_\beta\alpha\,k)\,\sin{}_\beta\alpha = \cos({}_\beta\alpha\,k)\cos{}_\beta\alpha \ . \tag{B2}$$

Received in editor's office : August 1, 2007
Approved for publishing : April 1, 2008

Note : The article is a modified extended English version of the contribution [6].