CRACK INITIATION CRITERIA FOR SINGULAR STRESS CONCENTRATIONS
Part II: Stability of Sharp and Bi-Material Notches

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The usual approach in assessing the stability of general singular stress concentrators developed and presented in the first part of this contribution is applied to configurations corresponding to bi-material notches. Two stability criteria based on the knowledge of the tangential stress and the strain energy density factor distribution in the vicinity of the bi-material notch are formulated. The critical stress for crack initiation and the initial crack propagation direction are calculated as a function of the geometry and material properties of the bi-material notch.

Key words: *fracture mechanics, sharp notch, bi-material notch, stability criteria, crack initiation, singular stress concentrator*

1. Introduction

This article attempts to reveal procedures of assessing conditions when a crack initiates at the tip of a bi-material notch. The existence of bi-material notches is connected with geometrical or material discontinuities. The general configuration of the bi-material notch is shown in Fig. 1. The geometry of the notch is given by the angles ω_1, ω_2 and the material properties characterised by the elastic constants E_1, ν_1, E_2, ν_2 corresponding to the materials 1 and 2.

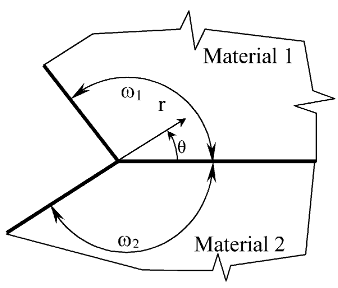

Fig.1: A bi-material notch characterised by the angles ω_1, ω_2, material properties of materials 1 and 2, and its polar coordinate system r, θ

Procedures suitable for estimation of crack initiation conditions in the vicinity of bi-material notches have not been quite known up to now. In the following a bi-material notch is considered as a special case of a general singular stress concentrator (GSSC). The object of the second part of this paper is to apply the general approach suggested and formulated in the

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first part [1] of the contribution to the assessment of the stability of bi-material notches. For that purpose two stability criteria based on the knowledge of the maximal tangential stress and the strain energy density factor distribution in the vicinity of the bi-material notch are formulated. The critical stress for crack initiation and the initial crack propagation direction are then calculated as a function of the geometry and material properties of the bi-material notch. The presented approach follows the basic idea of linear elastic fracture mechanics, i.e. the conditions of small-scale yielding are assumed and it has a phenomenological character. Furthermore, the ideal bi-material configuration is considered, i.e. with a sharp notch tip, step change of material properties through the interface and ideal adhesion at the interface.

2. Analysis of singular stress distribution

The stress field around a bi-material notch can be determined by means of the combination of analytical and numerical approaches, e.g. [2, 3, 4]. An analytical solution based on the real Airy stress function follows from the biharmonic partial differential equation

$$\Delta\Delta\Phi_m = 0 \ , \qquad \text{i.e.} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \Phi_m}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_m}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_m}{\partial \theta^2}\right) \ , \qquad (1)$$

where Δ is the Laplace operator. The bi-material notch consists of two domains, as shown in Fig. 1, addressed by $m = 1, 2$ with different material properties. The stress function Φ_m can be written in the form of an infinite series:

$$\Phi_m = \sum_{k=1}^{\infty} A_k \, r^{\lambda_k + 1} \, F_{km}(\theta, \lambda_k) \qquad (2)$$

often referred to as the Williams' eigenfunction expansion. A_k are unknown coefficients, λ_k are eigenvalues, F_{km} are eigenfunctions, and r, θ are the polar coordinates, see Fig. 1. Although the expression (2) is an infinite series, only terms singular with respect to the radial coordinate r are taken into account as the limit analytical solution.

By substituting Φ in (1), we achieve the fourth-order differential equation for the calculation of eigenfunctions F_m in the form (the subscript index k is omitted for the sake of simplicity):

$$F_m^{(4)} + 2\,(1 + \lambda^2)\, F_m'' + (1 - \lambda^2)^2\, F_m = 0 \ , \qquad (3)$$

where the prime denotes the derivative with respect to the angular coordinate θ. The characteristic equation corresponding to (3) has two complex-conjugate roots. The overall solution can then be expressed as the sum of two even cosine functions and two odd sine functions as

$$F_m(\theta, \lambda) = a_m \, \sin(\lambda + 1)\,\theta + b_m \, \cos(\lambda + 1)\,\theta - c_m \, \sin(\lambda - 1)\,\theta - d_m \, \cos(\lambda - 1)\,\theta \qquad (4)$$

where a_m, b_m, c_m, d_m are unknown constants. The polar stress components can now be obtained separately for each domain in terms of the eigenfunction F_m. By substituting (4) in the Williams' expansion (2) and using the relation for the stress components in the polar coordinates

$$\begin{aligned} \sigma_{m\mathrm{rr}} &= \frac{1}{r}\frac{\partial \Phi_m}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi_m}{\partial \theta^2} \ , \\ \sigma_{m\theta\theta} &= \frac{\partial^2 \Phi_m}{\partial r^2} \ , \\ \sigma_{m\mathrm{r}\theta} &= \frac{1}{r^2}\frac{\partial \Phi_m}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \Phi_m}{\partial r\,\partial \theta} \end{aligned} \qquad (5)$$

we then obtain the polar stress components in the form

$$\begin{aligned} \sigma_{m\mathrm{rr}} &= A\, r^{\lambda-1} \left(F_m'' + (\lambda+1)\, F_m\right) , \\ \sigma_{m\theta\theta} &= A\, r^{\lambda-1} \left(\lambda\, (\lambda+1)\, F_m\right) , \\ \sigma_{m\mathrm{r}\theta} &= A\, r^{\lambda-1} \left(-\lambda\, F_m'\right) . \end{aligned} \tag{6}$$

Therefore, the stress field has a singular behaviour with respect to the radial distance r for all allowed eingenvalues in the interval $0 < \lambda < 1$. By applying the Hooke's law to the set (6), the corresponding polar components of the displacements are obtained as

$$\begin{aligned} u_{m\mathrm{r}} &= A\, \frac{r^\lambda}{2\,\mu_m} \left\{ -(\lambda+1)\, F_m + \frac{1-\bar\nu_m}{\lambda} \left[F_m'' + (\lambda+1)^2\, F_m\right] \right\} , \\ u_{m\theta} &= A\, \frac{r^\lambda}{2\,\mu_m} \left\{ -F_m' + \frac{1-\bar\nu_m}{\lambda\,(\lambda-1)} \left[F_m''' + (\lambda+1)^2\, F_m'\right] \right\} , \end{aligned}$$

where $\bar\nu_m = \nu_m$ for plain strain, $\bar\nu_m = \nu_m/(1+\nu_m)$ for plain stress, and is Poisson's ratio of the material domain m.

The nine unknown parameters, i.e. a_m, b_m, c_m, d_m for $m = 1, 2$ and the eigenvalue λ have to be determined from the set of the boundary conditions corresponding to the particular problem.

Providing that the notch surfaces are traction-free, the following set of stress components along the notch faces has to be accomplished:

$$\sigma_{1\theta\theta}(r,\omega_1) = \sigma_{1\mathrm{r}\theta}(r,\omega_1) = \sigma_{2\theta\theta}(r,\,-\omega_2) = \sigma_{2\mathrm{r}\theta}(r,\,-\omega_2) = 0 \tag{7}$$

where the first subscript denotes the material to which a particular component corresponds. Considering that the interface between material 1 and 2 is perfectly bonded (ideal adhesion), the following set of equations for displacements must then be satisfied along the interface:

$$\begin{aligned} u_{1\mathrm{r}}(r,\theta=0) &= u_{2\mathrm{r}}(r,\theta=0) , \\ u_{1\theta}(r,\theta=0) &= u_{2\theta}(r,\theta=0) . \end{aligned} \tag{8}$$

For the same reason, the tractions are continuous across the interface, i.e.:

$$\begin{aligned} \sigma_{1\theta\theta}(r,\theta=0) &= \sigma_{2\theta\theta}(r,\theta=0) , \\ \sigma_{1\mathrm{r}\theta}(r,\theta=0) &= \sigma_{2\mathrm{r}\theta}(r,\theta=0) . \end{aligned} \tag{9}$$

These eight conditions form a homogeneous algebraic system with nine unknown parameters, namely λ and the eight coefficients creating the vector $\mathbf{x}$. The system $\mathbf{B}_{8\times8}(\lambda)\,\mathbf{x} = \mathbf{0}$, where $\mathbf{B}(\lambda)$ is the matrix of the system, is underdetermined. The existence of the nontrivial solution requires

$$\det \mathbf{B}(\lambda) = 0 , \tag{10}$$

from which the eigenvalues λ can be obtained. Theoretically, the infinite number of the eigenvalues can be found, but only λ in the interval $(0,1)$ corresponding to the singular terms are taken into account. Further, depending on the eigenvalue λ, the coefficients a_m, b_m, c_m, d_m for $m = 1, 2$ are determined. Note that only the proportions of seven coefficients to the eighth can be obtained in this way. The requested seven coefficients are quantified in terms of the eighth usually taken as 1. In the following, the value of the coefficient $d_2 = 1$. Under these conditions, the solution is determined except for the value A. This constant cannot be determined from the analytical approach and has to be evaluated numerically through analysis of the entire model with given boundary conditions.

2.1. Study of the eigenvalues for specific bi-material notches

As mentioned above, the stress components have a singular character for eigenvalues in the range $0 < \lambda < 1$. As an example the values λ_k have been determined for two basic geometrical configurations as shown in Fig. 2 and in dependence on the composite Dundurs' parameters α, β (11), see [6] for details,

$$\alpha = \frac{-\mu_1\,(\kappa_2+1)+\mu_2\,(\kappa_1+1)}{\mu_1\,(\kappa_2+1)+\mu_2\,(\kappa_1+1)}\;, \qquad \beta = \frac{-\mu_1\,(\kappa_2-1)+\mu_2\,(\kappa_1-1)}{\mu_1\,(\kappa_2+1)+\mu_2\,(\kappa_1+1)}\;, \tag{11}$$

where shear modulus $\mu_m = E_m/[2(1+\nu_m)]$ (E_m is Young's modulus), parameters $\kappa_m = (3-\nu_m)/(1+\nu_m)$ for the case of plane stress or $\kappa_m = 3-4\,\nu_m$ for plain strain ($m = 1,2$). It is $-1 \le \alpha \le 1$, $-0.5 \le \beta \le 0.5$ and the special case $\alpha = \beta = 0$ corresponds to a homogeneous body.

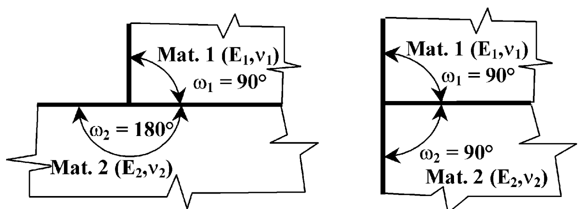

Fig.2: Two basic geometrical configurations of bi-material notches (materials 1 and 2) studied in this chapter

It follows from the previous analysis that the eigenvalues λ depend on the elastic constants of the two media and on the wedge angles, but do not depend on the body dimensions or on the external stresses:

$$\lambda = \lambda(\omega_1,\omega_2,\mu_1,\nu_1,\mu_2,\nu_2) = \lambda(\omega_1,\omega_2,\alpha,\beta)\;. \tag{12}$$

This enables us to discuss the dependence of the eigenvalues λ on the Dundurs' parameters α, β inside the Dundurs' parallelogram for fixed notch geometry, i.e. for fixed angles ω_1, ω_2. For this article, the eigenvalues λ have been determined numerically for the following (from the practical point of view) most important geometries (see Fig. 2):

a) Perpendicular face: $\omega_1 = 90°$, $\omega_2 = 180°$
b) Free edge: $\omega_1 = 90°$, $\omega_2 = 90°$

The corresponding Dundurs' parallelograms are shown in figures 3 and 4.

Figures 2a and 3 show that in the case of the perpendicular face the two real eigenvalues occur in most material combinations. The geometry corresponding to the free edge (Fig. 2b) leads only to one real eigenvalue λ_1 (see Fig. 4).

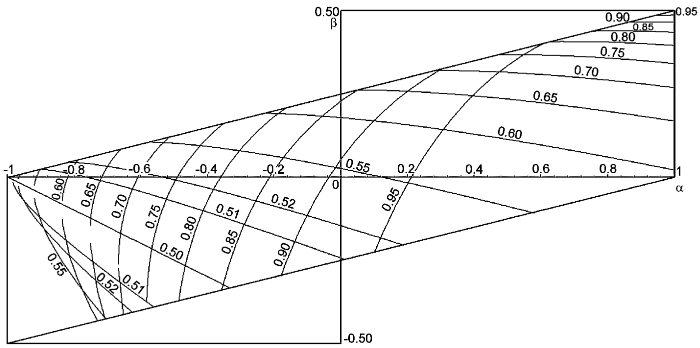

Fig.3: Eigenvalues λ_1 and λ_2 for the perpendicular face ($\omega_1 = 90°$, $\omega_2 = 180°$)

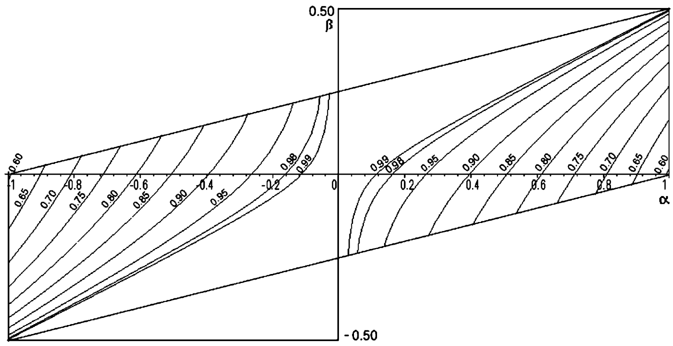

Fig.4: Eigenvalues λ_1 for a free edge ($\omega_1 = 90°$, $\omega_2 = 90°$)

2.2. Stress components

In the following, instead of the eigenvalues λ, the stress singularity exponents $p = 1 - \lambda$ are employed. In most of the geometrical and material configurations of a bi-material notch there are two real stress singularity exponents p_1 and p_2 in the interval $(0, 1)$. Contrary to a crack in homogeneous material, the exponents differ from 1/2 and, furthermore, each singular term includes both normal and shear mode of loading. Then the singular stress components can be written in the polar coordinates:

$$
\begin{aligned}
\sigma_{m\mathrm{rr}} &= \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}}\, r^{-p_k}\, F_{\mathrm{rr}km} \;, \\
\sigma_{m\theta\theta} &= \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}}\, r^{-p_k}\, F_{\theta\theta km} \;, \\
\sigma_{m\mathrm{r}\theta} &= \sum_{k=1}^{2} \frac{H_k}{\sqrt{2\pi}}\, r^{-p_k}\, F_{\mathrm{r}\theta km}
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
F_{\mathrm{rr}km} &= (2 - p_k)\,[-a_{mk} \sin((2 - p_k)\,\theta) - b_{mk}\, \cos((2 - p_k)\,\theta) + \\
&\qquad + 3\, c_{mk}\, \sin(-p_k\, \theta) + 3\, d_{mk}\, \cos(-p_k\, \theta)] \;, \\
F_{\theta\theta km} &= (p_k^2 - 3\, p_k + 2)\,[a_{mk} \sin((2 - p_k)\,\theta) + b_{mk}\, \cos((2 - p_k)\,\theta) + \\
&\qquad + c_{mk}\, \sin(-p_k\, \theta) + d_{mk}\, \cos(-p_k\, \theta)] \;,
\end{aligned}
$$

$$F_{r\theta km} = (2 - p_k)\,[-a_{mk}\cos((2 - p_k)\,\theta) + b_{mk}\,\sin((2 - p_k)\,\theta) + \\ + c_{mk}\,\cos(-p_k\,\theta) - d_{mk}\,\sin(-p_k\,\theta)]\ .$$

The subscript m differentiates materials 1 and 2 where the stresses are determined. The value H_k is the generalized stress intensity factor (GSIF), which has to be ascertained from the numerical solution of the studied geometry with given materials and boundary conditions, see chapter 2.3.

By substituting the same materials in the above formulas we get the relations for the stress components for a sharp notch in a homogeneous material. In this case the stress singularity exponents depend only on the notch opening angle and one or two exponents belong to the specific loading mode I or II.

2.3. Stress intensity factors

For the complete determination of the stress components, the stress intensity factors H_1 and H_2 have to be determined. One of the suitable ways used here is a direct method [3, 4, 5]. The method compares the results of some appropriate magnitude from a numerical solution with its analytical representation. The tangential stress $\sigma_{\theta\theta}$ is used here as the appropriate magnitude for comparison. If the stress distribution is described by a combination of H_1 and H_2, it is necessary to solve the system of two equations. To achieve this, the values of $\sigma_{\theta\theta}$ following from the finite element method are determined for two different angles θ_1, θ_2. Then knowing the analytical relations e.g. for $\sigma_{\theta\theta}$ (13) we solve the system of equations for H_1 and H_2 :

$$\begin{bmatrix} r^{-p_1}\,F_{\theta\theta 1m}(\theta = \theta_1) & r^{-p_2}\,F_{\theta\theta 2m}(\theta = \theta_1) \\ r^{-p_1}\,F_{\theta\theta 1m}(\theta = \theta_2) & r^{-p_2}\,F_{\theta\theta 2m}(\theta = \theta_2) \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} \sigma_{m\theta\theta}(r, \theta_1) \\ \sigma_{m\theta\theta}(r, \theta_2) \end{bmatrix}\ . \tag{14}$$

The valid values of GSIFs H_1 and H_2 are then determined by an extrapolation of the solutions (14) into $r = 0$. For details see the numerical example in chapter 4.

3. Criteria for crack initiation

The general principle of stability assessment of a general singular stress concentrator (GSSC) has been introduced in the first part of this article series [1]. The classic fracture mechanics approach of comparison of the stress intensity factor K_{I} with its critical value $K_{\mathrm{I\,crit}}$ (represented by fracture toughness K_{IC} or by the fatigue threshold value $K_{\mathrm{I\,th}}$) is generalized to the following relation :

$$H_k(\sigma_{\mathrm{appl}}) < H_{k\,\mathrm{crit}}(M_m)\ . \tag{15}$$

The value $H_k(\sigma_{\mathrm{appl}})$ follows from the numerical solution. Its critical value $H_{k\,\mathrm{crit}}$ depends on the critical material characteristic K_{IC} or $K_{\mathrm{I\,th}}$ and has to be deduced with help of a controlling variable L, see [1]. Note that values H_1 and H_2 are mutually dependent and their critical values H_{1C} and H_{2C} as well, see chapter 3.2.2.

In the following paragraphs the conditions for crack initiation are presented in five steps : choice of a suitable controlling variable, determination of the direction of crack initiation, estimation of the generalized stress intensity factors, determination of the critical value of the GSIFs and determination of critical applied stress.

3.1. Choice and introduction of a suitable controlling variable

The controlling variable L needs to have a clear and identical physical meaning in the case of assessing both a crack in a homogeneous material and a bi-material notch [1]. The particularities of the bi-material notches are:

- The inherently combined mode of loading that follows from the stress distribution (13),
- The existence of two singularities in most of the geometrical and material configurations.

In order to demonstrate the procedures of stability condition suggestions, two controlling variables L are considered: (1) the mean value of the stress component $\sigma_{\theta\theta}$ and (2) the mean value of the strain energy density factor Σ.

3.2. Criterion of the mean value of tangential stress

This criterion is based on monitoring tangential stress around the notch tip. The mean value of $\sigma_{\theta\theta}$ component over a certain distance d

$$\bar{\sigma}_{\theta\theta}(\theta) = \frac{1}{d} \int\limits_0^d \sigma_{\theta\theta}(r,\theta)\,\mathrm{d}r \tag{16}$$

determined in dependence on the polar angle θ is considered as the controlling variable L.

By analogy with cracks in the homogeneous case it is supposed that the crack at the bi-material notch tip is initiated in the direction θ_0 where $\bar{\sigma}_{\theta\theta}(\theta)$ has its maximum. Further, it is assumed that the crack is initiated when $\bar{\sigma}_{\theta\theta}(\theta_0)$ reaches its critical value $\bar{\sigma}_{\theta\theta\,\mathrm{C}}(\theta_0)$ that is ascertained for a crack in homogeneous media. The distance d has to be chosen in dependence on the mechanism of a rupture, e.g. as a dimension of a plastic zone or as a size of material grain.

3.2.1. Crack initiation direction

The potential direction of crack initiation is determined from the maximum of the mean value of tangential stress in both materials. The following two conditions have to be satisfied:

$$\left(\frac{\partial \bar{\sigma}_{m\theta\theta}}{\partial \theta}\right)_{\theta_0} = 0\;, \qquad \left(\frac{\partial^2 \bar{\sigma}_{m\theta\theta}}{\partial \theta^2}\right)_{\theta_0} < 0\;. \tag{17}$$

Using (13) and (16) the average tangential stress for a bi-material wedge can be expressed as:

$$\bar{\sigma}_{\theta\theta m} = \frac{H_1}{\sqrt{2\pi}}\,\frac{d^{-p_1}}{1-p_1}\,F_{\theta\theta 1m} + \frac{H_2}{\sqrt{2\pi}}\,\frac{d^{-p_2}}{1-p_2}\,F_{\theta\theta 2m} \tag{18}$$

and for its first derivation it follows:

$$\frac{d^{-p_1}}{1-p_1}\,\frac{\partial F_{\theta\theta 1m}}{\partial \theta} + \frac{H_2}{H_1}\,\frac{d^{-p_2}}{1-p_2}\,\frac{\partial F_{\theta\theta 2m}}{\partial \theta} = 0 \tag{19}$$

where:

$$\frac{\partial F_{\theta\theta km}}{\partial \theta} = (p_k^2 - 3\,p_k + 2)\,\{(2-p_k)\,[a_{mk}\,\cos((2-p_k)\,\theta) - b_{mk}\,\sin((2-p_k)\,\theta)] - \\ - p_k\,[c_{mk}\,\cos(p_k\,\theta) + d_{mk}\,\sin(p_k\,\theta)]\}$$

for $k = 1, 2$.

It is obvious that by inserting the ratio of the values H_2/H_1 (obtained from the numerical solution) into the relation (19), we obtain a simple equation for the value of θ_0. The maximum of $\sigma_{m\theta\theta}$ can exist in both material 1 in the interval $(0, \omega_1)$ and material 2 in the interval $(-\omega_2, 0)$. If there are more than one direction of possible crack initiation, it is necessary to consider all of them.

3.2.2. Stability criterion suggestion

Here we present the suggestion of the stability criterion based on the average stress calculated across a distance d from the wedge tip. The value of the average stress $\bar{\sigma}_{\theta\theta}$ corresponding to the bi-material wedge is calculated for the direction of θ_0 and it is compared with the critical stress $\bar{\sigma}_{\theta\theta\,\mathrm{C}}$ corresponding to the crack [7].

For a crack in homogeneous material under mode I of loading we obtain (the direction of assumed crack propagation $\theta_0 = 0$):

$$\bar{\sigma}_{\theta\theta\,\mathrm{C}} = \frac{2\,K_{\mathrm{IC}}}{\sqrt{2\pi\,d}}\ . \tag{20}$$

To find the relation between $H_k(\sigma_{\mathrm{appl}})$ and $H_{k\,\mathrm{crit}}(M_m)$, let us consider the fact that the ratio of the values H_1 and H_2 is constant for a given bi-material configuration and boundary conditions and does not depend on the value of the applied stress σ_{appl} and it holds

$$\Gamma_{21} = \frac{H_2}{H_1} = \frac{H_{2\mathrm{C}}}{H_{1\mathrm{C}}}\ , \quad [\mathrm{m}^{p_2-p_1}]\ .$$

This assumption is justified because when changing the value of the applied stress, only the absolute values of GSIFs change, but their ratio is constant even for the critical values $H_{2\mathrm{C}}/H_{1\mathrm{C}}$. The ratio has no physical meaning, it just represents the contribution of particular singular terms to the stress distribution. Inserting the ratio Γ_{21} and the critical value $H_{1\mathrm{C}}$ into the relation (18) we get the critical value of the average tangential stress for a bi-material wedge. Following the assumption of the same mechanism of a rupture in both cases (crack and notch) we can compare it with the relations for a crack (20) and obtain an expression for $H_{1\mathrm{C}}$ value:

$$H_{1\mathrm{C}} = \frac{2\,K_{\mathrm{IC}}}{\dfrac{d^{\frac{1}{2}-p_1}}{1-p_1}\,F_{\theta\theta 1m}(\theta_0) + \Gamma_{21}\,\dfrac{d^{\frac{1}{2}-p_2}}{1-p_2}\,F_{\theta\theta 2m}(\theta_0)}\ . \tag{21}$$

It is evident that the critical value $H_{1\mathrm{C}}$ depends on the value of fracture toughness K_{IC} that is a material characteristic. The critical values are compared for the directions θ_0 of assumed crack initiation that is in case of bi-material notch ascertained from maximal tangential stress criterion. Because of that fact, the normal loading mode is predominant. Thus the comparison with crack propagation characterized by fracture toughness K_{IC} is justified. The integrating distance d has to be chosen according to the mechanism of the rupture, e.g. for a cleavage fracture it can be set between 2–5× grain size of the material.

Then the stability criterion can be suggested in the following form:

$$H_1(\sigma_{\mathrm{appl}}) < H_{1\mathrm{C}}(K_{\mathrm{IC}})\ . \tag{22}$$

The crack is not initiated in the tip of a bi-material notch if the value H_1 of GSIF is lower than its critical value H_{1C}. The value H_1 is determined from a numerical solution of a given bi-material configuration with given boundary conditions, geometry, and material properties of the bi-material wedge. The critical value H_{1C} is given by the relation (21).

3.3. Strain energy density factor criterion

Similarly, the stability conditions can be derived via the mean value of the strain energy density factor.

For a crack in homogeneous material, $S(r,\theta)$, the strain energy density factor (SEDF), is defined as [8]:

$$S(\theta) = r\,\frac{\mathrm{d}W}{\mathrm{d}V} = r\int_0^{\varepsilon} \sigma\,\mathrm{d}\varepsilon \ , \tag{23}$$

where $\mathrm{d}W/\mathrm{d}V$ is the corresponding strain energy density.

Analogically, a generalized strain energy density factor (GSEDF) for a bi-material notch, $\Sigma(r,\theta)$, can be set up as:

$$\Sigma(r,\theta) = r\,\frac{\mathrm{d}W}{\mathrm{d}V} = r\int_0^{\varepsilon} \sigma\,\mathrm{d}\varepsilon \ , \tag{24}$$

where the corresponding stress components are given by (13).

Considering the cases of plane strain or plane stress, it becomes:

$$\Sigma_m = r\,\frac{2\,\sigma_{m\theta\theta}\,\sigma_{m\mathrm{rr}}\,(k_m-1) + (\sigma^2_{m\theta\theta} + \sigma^2_{m\mathrm{rr}})\,(k_m+1) + 4\,\sigma^2_{m\mathrm{r}\theta}}{8\,\mu_m} \ . \tag{25}$$

Here $k_m = (1-\nu_m)/(1+\nu_m)$ holds for plane stress and $k_m = (1-2\,\nu_m)$ for plane strain, μ_m is the shear modulus and ν_m is the Poisson's ratio of material $m = 1, 2$. Substituting the relations (13) in (25) we can obtain the expression for the distribution of the generalized strain energy density factor in the vicinity of the bi-material notch tip:

$$\Sigma_m = \frac{1}{8\,\mu_m}\,\frac{H_1^2}{2\pi}\,(r^{1-2p_1}\,U_{1m} + r^{1-2p_2}\,\Gamma^2_{21}\,U_{2m} + r^{1-p_1-p_2}\,2\,\Gamma_{21}\,U_{12m}) \tag{26}$$

where

$$\begin{aligned}
U_{1m} &= (F^2_{\mathrm{rr}1m} + F^2_{\theta\theta 1m})(k_m+1) + 4\,F^2_{\mathrm{r}\theta 1m} + 2\,F_{\theta\theta 1m}\,F_{\mathrm{rr}1m}\,(k_m-1) \ , \\
U_{2m} &= (F^2_{\mathrm{rr}2m} + F^2_{\theta\theta 2m})(k_m+1) + 4\,F^2_{\mathrm{r}\theta 2m} + 2\,F_{\theta\theta 2m}\,F_{\mathrm{rr}2m}\,(k_m-1) \ , \\
U_{12m} &= (F_{\mathrm{rr}1m}\,F_{\mathrm{rr}2m} + F_{\theta\theta 1m}\,F_{\theta\theta 2m})(k_m+1) + 4\,F_{\mathrm{r}\theta 1m}\,F_{\mathrm{r}\theta 2m} + \\
&\qquad + (F_{\theta\theta 1m}\,F_{\mathrm{rr}2m} + F_{\theta\theta 2m}\,F_{\mathrm{rr}1m})\,(k_m-1) \ .
\end{aligned}$$

While in the case of a crack in homogeneous media Sih's SEDF does not depend on the radial coordinate r, see e.g. [8, 9], in the case of a homogeneous or bi-material notch or in the case of a crack with its tip at a bi-material interface [5] the value of GSEDF depends on the polar coordinate r, i.e. $\Sigma = \Sigma(r,\theta)$.

In the following part, the mean value of generalized SEDF over a certain distance d from the notch tip will be considered as a suitable controlling variable, i.e. $L = \bar{\Sigma}(d, \theta)$. For material m it is

$$\bar{\Sigma}_m = \frac{1}{d} \int_0^d \Sigma_m \, \mathrm{d}r = \\ = \frac{H_1^2}{16\,\mu_m\,\pi} \left(\frac{d^{1-2p_1}}{2-2p_1}\, U_{1m} + \frac{d^{1-2p_2}}{2-2p_2}\, \Gamma_{21}^2\, U_{2m} + \frac{d^{1-p_1-p_2}}{2-p_1-p_2}\, 2\,\Gamma_{21}\, U_{12m} \right) . \tag{27}$$

The integration distance d enters the calculations as a structural parameter or a parameter related to the mechanism of rupture. Note that in the case of a crack in homogeneous material where S does not depend on r, the mean value of the strain energy density factor $\bar{S}(\theta) = S(\theta)$.

3.3.1. Crack initiation direction

Following the basic assumption of the SEDF theory [8, 9, 10], the crack propagation direction θ_0 is identical with the direction of the local minimum of the strain energy density $S(\theta_0)$. By analogy it is assumed here that in the case of a bi-material notch the potential direction of crack initiation is identical with the direction of the minimum of the mean value of the generalized strain energy density factor $\bar{\Sigma}$.

To find its minimum in material $m = 1, 2$, two conditions have to be determined:

$$\frac{\partial \bar{\Sigma}_m}{\partial \theta} = 0 \ , \qquad \frac{\partial^2 \bar{\Sigma}_m}{\partial \theta^2} < 0 \ . \tag{28}$$

Differentiating the relation (27), the first derivation from (28) can be expressed in the form:

$$\frac{d^{2-2p_1}}{2-2p_1}\,\frac{\partial U_{1m}}{\partial \theta} + \frac{d^{2-2p_2}}{2-2p_2}\,\Gamma_{21}^2\,\frac{\partial U_{2m}}{\partial \theta} + \frac{d^{2-p_1-p_2}}{2-p_1-p_2}\,2\,\Gamma_{21}\,\frac{\partial U_{12m}}{\partial \theta} = 0 \ . \tag{29}$$

It is evident that the direction of potential crack initiation again does not depend on the absolute value of the GSIFs, but depends only on their ratio Γ_{21}. The ratio Γ_{21} can be taken from the direct methods based on tangential stress, as shown in chapter 2.3. From the two possible solutions (29) one implies the minimum of the mean value of SEDF and it satisfies the condition of positive second derivative (28).

3.3.2. Stability criterion suggestion

Like in the previous criterion (22) the crack will not be initiated from the bi-material notch tip if under given conditions the value H_1 is less than its critical value $H_{1\mathrm{C}}$ (generalized fracture toughness in the case of brittle fracture or generalized threshold value $H_{1\mathrm{th}}$ in the case of fatigue loading), i.e.

$$H_1(\sigma_{\mathrm{appl}}) < H_{1\mathrm{th}}(K_{\mathrm{Ith}}) \ .$$

Proceeding from the suggestion of the same mechanism of rupture and consequently the same value of average generalized SEDF corresponding to critical conditions in the case of a crack and a bi-material notch $\bar{\Sigma}_{m\,\mathrm{C}} = S_{m\,\mathrm{C}}$, and the generalized critical values of GSIF are given as follows:

$$H_{1\mathrm{th},m} = 2\,K_{\mathrm{I\,th},m} \sqrt{\frac{k_m}{\frac{d^{1-2p_1}}{2-2p_1}\,U_{1m} + \frac{d^{1-2p_2}}{2-2p_2}\,\Gamma_{21}^2\,U_{2m} + \frac{d^{1-p_1-p_2}}{2-p_1-p_2}\,2\,\Gamma_{21}\,U_{12m}}} \tag{30}$$

where $K_{\mathrm{I\,th},m}$ is the threshold value for material m.

3.4. Stability curve for a bi-material notch and the critical applied stress

The calculated value of the critical or threshold generalized stress intensity factor holds only for a particular value of the ratio H_2/H_1, i.e. for concrete boundary loading conditions. If the direction of the external loading is modified, the ratio of GSIF's changes, too. Generally, the graph H_1 vs. Γ_{21} (Fig. 5) can be drawn where the values $H_{1\mathrm{C},m}$ or $H_{1\mathrm{th},m}$ corresponding to eq. (21) or (30) and calculated for a varying ratio Γ_{21} form a stability curve. Note that one stability curve corresponds to each combination of the stress singularity exponents p_1 and p_2, i.e. to the special material combination and geometry of the bi-material notch.

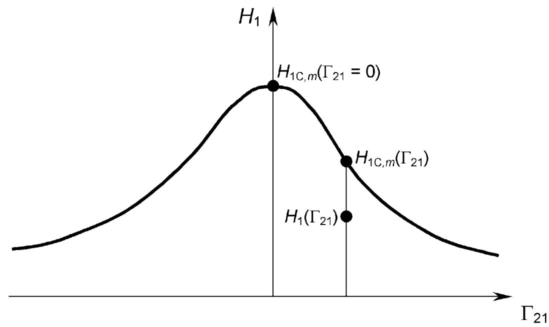

Fig.5: The stability curve for a combined loading mode of a bi-material wedge

The stability curve drawn in H_1 vs. Γ_{21} plot divides the area of the graph into a part where a crack is initiated and a part where the calculated values of $H_1(\Gamma_{21})$ below the curve guarantee the stability of the bi-material notch, see Fig. 5. Note that point on the vertical axe corresponding to $H_2 = 0$ hold for stress distribution with one singularity. The stability curve is created by points with coordinates $[H_{1\mathrm{C},m}, \Gamma_{21}]$ or $[H_{1\mathrm{th},m}, \Gamma_{21}]$ corresponding to the critical or threshold values of GSIF calculated for particular ratio Γ_{21}.

When assessing a concrete bi-material notch, several potential crack initiation angles can occur. Although the values of $H_{1\mathrm{C},m}$ or $H_{1\mathrm{th},m}$ are calculated in all possible directions $\theta_{0,m}$ in both materials, we take only the least value and we name it $H_{1\mathrm{crit}}$ in order to interpret the suggested stability condition in terms of the critical applied stress.

The critical applied stress can be formulated as:

$$\sigma_{\mathrm{crit}} = \sigma_{\mathrm{appl}}\,\frac{H_{1\mathrm{crit}}}{H_1(\sigma_{\mathrm{appl}})}\ . \tag{31}$$

Where σ_{appl} in the relation (31) is the stress applied in the numerical solution for the value H_1. The crack will not be initiated in the bi-material wedge tip if the applied stress is lower than the critical stress:

$$\sigma_{\mathrm{appl}} < \sigma_{\mathrm{crit}} \ . \tag{32}$$

4. Numerical example

For a numerical example a rectangular notch (fig. 6) is assessed. Geometry is given by the angles $\omega_1 = 90°$, $\omega_2 = 180°$. Two material configurations are taken into account. The bi-material notch with the following characteristics $E_1 = 1{\times}10^5$ MPa, $E_2 = 2{\times}10^5$ MPa, $\nu_1 = \nu_2 = 0.3$, $K_{\mathrm{Ith}} = 8\,\mathrm{MPa\,m}^{1/2}$ and the homogeneous notch with $E_1 = E_2 = 2{\times}10^5$ MPa, $\nu_1 = \nu_2 = 0.3$, $K_{\mathrm{Ith}} = 8\,\mathrm{MPa\,m}^{1/2}$. The stress singularity exponents follow from the geometry and material characteristics. In the bi-material case they are $p_1 = 0.4142$ and $p_2 = 0.0406$ while for the homogeneous body they are $p_1 = 0.4555$ and $p_2 = 0.0915$. The conditions of plane strain are considered.

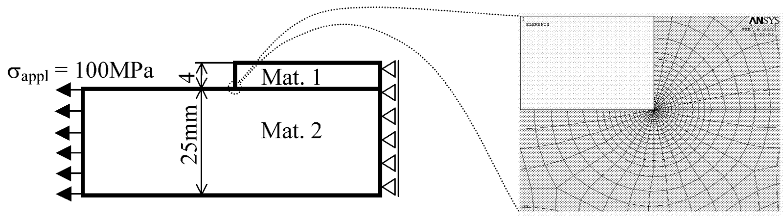

Fig.6: Rectangular bi-material wedge used in the numerical example, a detail of a FEM mesh

The finite element system ANSYS is used for the calculations. The generalized stress intensity factors H_1 and H_2 are gained from the direct method (14) by extrapolation into $r = 0$, see the estimation of H_1 in Fig. 7, the extrapolation in the case of H_2 is carried out in a similar way. Table 1 shows the values H_1, H_2, crack initiation angles θ_0, and critical values of GSIF $H_{1\mathrm{crit}} = \{H_{1\mathrm{C}}, H_{1\mathrm{th}}\}$ ascertained from the two suggested stability criteria: (i) criterion of the mean value of tangential stress, (ii) criterion of the mean value of the strain energy density factor (chapter 3.3). The length parameter d appearing in the criteria was chosen $d = 0.4$ mm. Finally, the critical applied stresses corresponding to initiation loading are stated in the table 1 as well. All the results are computed for both material configurations – bi-material and homogeneous notches.

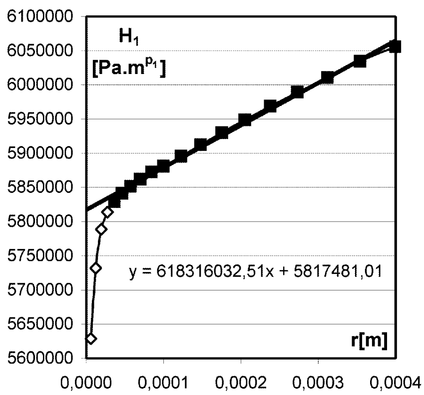

Fig.7: Extrapolation of H_1 value into the notch ($r = 0$)

		Bi-material notch		Homogeneous body	
Criterion of:		mean $\sigma_{\theta\theta}$	mean SEDF	mean $\sigma_{\theta\theta}$	mean SEDF
H_1	[MPa m^{p_1}]	5.817		5.464	
H_2	[MPa m^{p_1}]	−2.013		−9.506	
θ_0	[°]	−75.0	−69.3	−69.8	−65.4
$H_{1\mathrm{th}}$	[MPa m^{p_1}]	10.710	10.793	7.930	8.508
σ_{crit}	[MPa]	184.10	185.53	145.13	155.69

Tab.1: Results of numerical analysis of the notch in the bi-material and homogeneous body

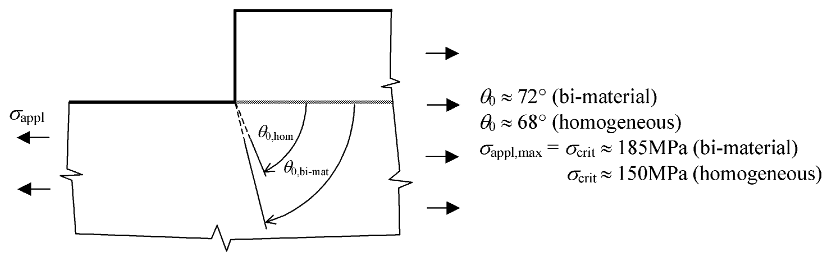

Fig.8: Crack initiation directions θ_0 and maximal applied stresses σ_{crit} corresponding to the case of a bi-material or homogeneous notch

As the Table 1 shows, both methods produce similar results for all of the studied magnitudes. In the case of the bi-material configuration the crack initiation angle $\theta_0 \approx 72°$ is greater than in the case of the homogeneous notch, $\theta_0 \approx 68°$. The difference in the results is similar as in case of a crack in a homogeneous media. Note that in case of GSEDF criterion the Poisson's ratio is covered into calculations and also the cases of plane strain and plain stress can be differentiated.

For the studied bi-material configuration $E_1 = E_2/2$ the value of the critical applied stress requires greater external loading for crack initiation than in the case of the notch in homogeneous material $E = E_2$, see Fig. 8.

5. Conclusions

On the basis of knowledge of the stress distribution around the bi-material notch two criteria of stability based on the mean value of tangential stress and the mean value of the strain energy density factor were suggested. The first one is generalisation of the maximum tangential stress criteria [12] and can be used in the case of damage caused by brittle fracture. The second one generalizes criterion of SEDF [8, 9] and is advantageous in the case of cyclic loading. Relations for critical or threshold values of generalized stress intensity factors were derived by means of the two controlling magnitudes and were used for the calculation of the critical applied stresses. The suggested procedures are illustrated in numerical examples. The model of a bi-material notch presented in the article can generally be used for assessing stability conditions of general singular material and geometrical discontinuities.

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