MATHEMATICAL MODEL OF PSEUDOINTERACTIVE SET: 1D BODY ON NON-LINEAR SUBSOIL I. THEORETICAL ASPECTS

Jiří V. Horák*, Horymír Netuka*

Mathematical model of pseudointeractive set of an elastic body (beam, plate) and subsoil for a special class of linear and non-linear response functions has been introduced. Brief review of the fundamental mathematical apparatus used for the analysis of the resulting non-linear boundary value problem has been given and discussed. Some of the typical statements concerning solvability of the model problem having form of linear and non-linear coercive and semi-coercive variational equation or inequality have been formulated, including sketches and remarks to their proofs. The emphasis has been focused on the semi-coercive case representing the typical problem of a free (unattached) body lying on a 'unilateral' subsoil defined by non-linear response function. Extra conditions of solvability have been formulated in the semi-coercive cases. The decomposition of Sobolev function space of kinematically admissible displacements into a cone of rigid displacement and its negative polar cone of displacements with non-zero deformation energy has been used to prove the existence of the solution in semi-coercive cases. The generalization of the form of the response function representing behavior of subsoil model has been also mentioned.

Key words : *unilateral subsoil, non-linear boundary value problems, semi-coercivity, decomposition of Hilbert space*

1. Introduction

In the study we present a brief introduction to the issue of mathematical modeling of a special class of *interactive sets* of the type *a free elastic body* and *subsoil*, see Fig. 1, for example. In the *Part I. Theoretical aspects*, we give concise review of the mathematical foundation concerning methods of solving of the continuous formulation of the problem while in the *Part II. Computational Aspects and Applications*, we define discrete model and give numerical methods used for building of computational model and some illustrative examples and applications.

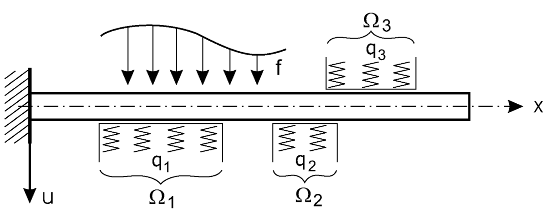

Fig.1: Scheme of beam–subsoil mechanical model

* Doc. RNDr. Ing. J. V. Horák, CSc., KMAaAM PřF UP, tř. Svobody 26, 746 11 Olomouc
* RNDr. H. Netuka, Ph.D., KMAaAM PřF UP, tř. Svobody 26, 746 11 Olomouc

For the sake of brevity, our study has been restricted only to the basic class of 'pseudocontact' problems of beams or plates on the 'unilateral' subsoil of Winkler's or piecewise Winkler's type, see Fig. 2 for considered class of subsoil models. Thus, instead of using standard model of the classical contact problem (see [3], [7], for example) between two or more elastic mutually non-penetrated bodies (beam or plate and different parts of subsoil), we are defining a quite different approach where the behavior of the subsoil in the mathematical model has been described via suitable extra *response* function $s = \hat{s}(q, u, D^{(k)}u, \theta, \dots)$ depending on the coefficient q of subsoil stiffness (surrounding material), displacement u, its derivatives $D^{(k)}u$, $k = 0, 1, \dots$, temperature θ, and so on, instead of considering another body representing subsoil and its reaction; for the case of circular plate omitted here, see [14]. To emphasize and illustrate mainly the crucial ideas and approach, we have been discussing only 1D beam (plate strip) problems in the paper, 2D plate models are omitted due to technical reasons and simplicity of the presentation.

2. Mathematical and mechanical concepts

In this section we are going to make readers acquainted with some main ideas and concepts concerning mechanical problems mentioned above, and with methods and tools of analysis and mathematical model design.

2.1. Response function

One of the key concept of an effective computational model design is installation of an additional 'auxiliary' function which represents in our definition of the mathematical model the response or the stiffness of the subsoil material; because of simplicity we have been concentrating in this paper exclusively on the functions $\hat{s}$ depending only on the deflection function $u = \hat{u}(x)$, $x \in \Omega = (0, l)$ of the beam (plate strip), thus $s(x) = \hat{s}(u)(x)$, i.e. we have been discussing only Winkler like one-parametrical models of subsoil. Furthermore, the response $\hat{s}$ is in all introduced cases supposed to be a convex and piecewise linear function. Generally, it can even be a multi-valued function and two corresponding basic cases are as follows. The first case can appear when the last (it can be even first if it is the only one) 'layer' of the subsoil is created by the perfect rigid subsoil, see Fig. 2 and Fig. 6; corresponding mathematical model has then the form of variational inequality of the first kind, see [6], and the second case can appear when $\hat{s}$ is the multi-valued function representing a subgradient of a given potential or superpotential, see [7], [8], [13], for example.

2.2. Mathematical formulation preliminaries

Some degrees of freedom of the unattached body (beam, plate strip) laying on the elastic foundation should be left in the mechanical model of the pseudocontact set for better description and more realistic approximation of the genuine situation; corresponding mathematical and computational model will be then only semi-coercive one, therefore corresponding stiffness matrix $\mathbf{K}$ (arising in FE model after discretization, see *Part II. Computational Aspects and Applications*) will be singular and only positive semidefinite (assuming that Neumann boundary conditions, i.e. free ends are given).

In all such cases we have to formulate extra necessary and sufficient conditions to guarantee the existence or uniqueness of the solution of the mathematical formulation of the

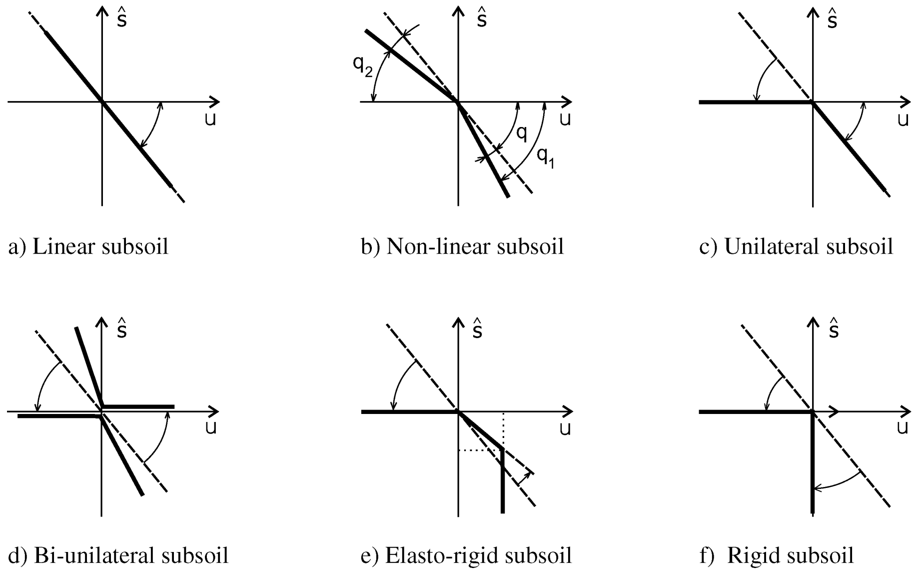

Fig.2: Graphs of response functions $\hat{s}$

problem (concerning all continuous, discrete, algebraic as well as computational models), see [11], [15], [18] and [19], for example.

The correct mathematical formulation of the introduced mechanical model design of pseudocontact problem has the form of either linear or non-linear boundary value problem, thus we have to use the terms of variational, weak or generalized solution to define the formulation properly, see [3], [16] for example. These are specified by the means of the terms of Hilbert spaces $\boldsymbol{H}$ of functions with finite energy, see [4], sets of kinematically admissible displacements $\mathcal{V} \subset \boldsymbol{H}$ and convex sets of restrictions $\mathcal{K} \subset \mathcal{V}$, see [17]. Furthermore, to succeed we have to use the apparatus for solving of the non-linear variational equations and inequalities, see [5], [6]. In the symmetry case, equivalent problem formulation can occur in the form of more physically and mechanically plausible definition, that is in the form of a minimization problem of the functional of the total potential energy of the system $\boldsymbol{J}$ (i.e. potential or superpotential) over the set of kinematically admissible deflections $\mathcal{V}$ or $\mathcal{K}$, see [6], [7], [17].

2.3. Existence of a solution

The outline of the proof of the existence of the model problem solution has been given for general (including semi-coercive) situation: we have been using the orthogonal decomposition of the linear space $\mathcal{V}$ of the kinematically admissible displacements into convex cone $\mathcal{K}_{\mathcal{V}}$ of null energy functions of the system and its negative polar cone $\mathcal{K}_{\mathcal{V}}^{\ominus}$, i.e. $\mathcal{V} = \mathcal{K}_{\mathcal{V}} \oplus \mathcal{K}_{\mathcal{V}}^{\ominus}$. Then $\mathcal{K}_{\mathcal{V}} \subset \mathcal{K}$ and $\mathcal{K}$ is a convex cone of all kinematically admissible *rigid* deflections, while $\mathcal{K}_{\mathcal{V}}^{\ominus}$ is a corresponding negative polar cone of 'real' deflections with non-zero deformation energy.

On the cone $\mathcal{K}_{\mathcal{V}}^{\ominus}$ and by the means of the conditions of solvability, we prove the coerciveness of the corresponding potential or the ellipticity of the corresponding form. From the mechanical point of view it means, we are able to force out (through suitable but realistic character of given load and/or given boundary conditions) the existence of the non-zero (active) area of contact zone on the feasible foundation, i.e. put on the set at equilibrium.

The elementary example illustrating the character of semi-coercivity and non-linearity of the model problem and its main complications, which have to be overcome in the design of computational model as well as practical examples illustrating some of the computational aspects will be given in *Part II* (further details concerning the model problem approximation, FEM discretization and algorithm of the numerical solution can be found in [1], [7], [19], for example).

2.4. Aim of this work

The aim of the presented paper is an illustration of the fundamental differences in the design and solution methods of the mathematical models of the problem class represented by linear and non-linear variational equations (or, in limit cases, by variational inequalities). The models correspond to the problems with monotonous operators defined either on Hilbert spaces $\mathcal{V} \subset \boldsymbol{H}$ or over their subsets of additional restrictions and constraints $\mathcal{K}$ (for corresponding example of layered subsoil with perfectly rigid foundation in depth $\sum_i L_i$ see Fig. 6).

Thus, we are going to describe the methodology of the solution design for problems range between the two very special *limit* cases: from *linear* response functions $s = \hat{s}(u)$ (corresponding to classical Winkler's linear model of 'bilateral' subsoil), through *piecewise linear* functions (non-linear 'bilateral' or 'unilateral' subsoil) up to *multi-value* functions (Signorini's model representing 'perfectly rigid' subsoil, i.e. limit case of the stiffness coefficient $q \to +\infty$ in Fig. 5 or Fig. 2f). The problems range can be easily illustrated and understand by the means of simple manipulations with the values of the stiffness coefficients of the bilateral subsoil q_i, $i = 1, 2$ in the definition of the corresponding response function, see Fig. 2, Fig. 4 and Fig. 5.

Note that these two limit cases, Winkler's model of elastic subsoil defined by linear response function, see Fig. 2a, and Signorini's model of unilateral rigid subsoil 'represented' by special multi-valued response function, see Fig. 2f, demarcate class of suitable, very useful models of subsoils, but none of them alone is very convenient for the practical needs, because they both represent quite non-realistic behavior of subsoil, either the issue of tensile stresses on some parts of 'contact' surface of subsoil or zero deflections of the body (perfectly rigid subsoil).

From the mathematical point of view we have been discussing the solvability of one sufficiently broad class of the problems having different forms: starting from linear variational equations through non-linear equations up to variational inequalities, all of them in coercive as well as in semi-coercive cases. All the presented results can also be used as a basic mathematical apparatus for generalization and design of new mathematical model class of the pseudointeractive sets within the framework of the theory of coupled thermoelasticity (thermoelastic beams, plates on subsoil), see [9]. In such a case, and by the means of Rothe method of discretization in time, we can prove the solvability of the corresponding mathematical model of the evolutional problem via its approximation by the sequence of stationary problems, solvability of each one can be proved by the approach introduced in this paper, see also [10].

3. Linear model of subsoil

The first limit case of the defined class of subsoil models, the standard applied model of Winkler's bilateral subsoil (note the response function of subsoil has the form $\hat{s}(u)(x) = q(x)\,u(x)$, see Fig. 2a) has, as it was mentioned above, very exceptional position between all models introduced here.

On the one hand, employing this classical and historical approach is very 'pleasant', because the mathematical attitude of the model is quite simple, and it leads to *a linear problem with unique solution* (in both continuous and discrete case) regardless chosen type of classical or non-classical *linear* boundary conditions (corresponding stiffness matrix $\mathbf{K}$ in FE model is regular and positive definite, for example).

To prove this it is enough to assume that 'an active' part of subsoil physically exists at least on a 'non-zero' area of the beam or plate, i.e. $\exists i \in I : \tilde{\Omega}_i \subset \Omega$, $\mu(\tilde{\Omega}_i) > 0$ and $q(x) \geq q_0 > 0$ for a.e. $x \in \tilde{\Omega}_i$ (see Fig. 1), where $\tilde{\Omega} \subset \Omega$, $\tilde{\Omega} = \cup_{j \in I} \tilde{\Omega}_j$, (where $\Omega = (0, l)$ for beam) and $\mu = \mu(x)$ is $\mathcal{L}$ – measure). Then, with this realistic assumption, we can use standard approach [5], [6], [16], see below.

On the other hand, the mechanical attitude of this 'two-sided' or 'fixed' (bilateral) model is rather problematic, because it can somewhere induce physically non-real tensile stresses (body and subsoil are bonded in this model, i.e. body behaves like it was firmly embedded in elastic or rigid environment).

Furthermore, it is easy to prove that any of the following non-linear models belonging to the defined class of subsoil models can be directly derived from this linear one by means of an appropriate manipulation with the definition of response function $\hat{s}(.)$, see Fig. 2 for the conception and illustration; thus standard linear Winkler model Fig. 3 ($\mathbf{j}$ is the potential and $\mathbf{R}$ the reaction of the subsoil corresponding to the deflection $\mathbf{u}$) is a special case of general non-linear model, Fig. 4.

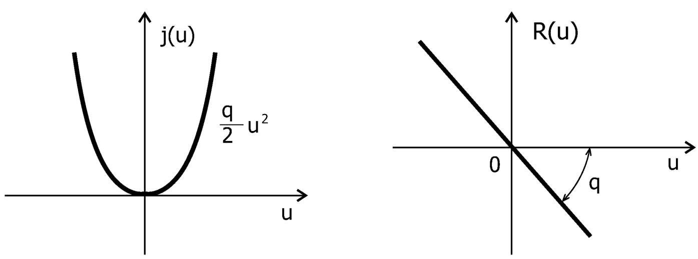

Fig.3: Scheme of linear Winkler subsoil model : potential and response function

To be brief, we concentrate in the remaining paragraphs only on the beam problem with free ends on Winkler bilateral subsoil; variational formulation of the model problem has then the form of minimization problem of the potential energy functional over the linear space of kinematically admissible deflections $\mathcal{V} = H^2(\Omega)$, i.e. *model problem* reads

$$u \in H^2(\Omega) : \quad \boldsymbol{J}(u) \leq \boldsymbol{J}(v) \qquad \forall v \in H^2(\Omega) \tag{1}$$

where

$$\boldsymbol{J}(v) = \frac{1}{2}\,\boldsymbol{a}(v,v) + \frac{1}{2}\boldsymbol{s}(v,v) - \mathcal{F}(v) - \sum_i F_i\,\delta_{x_i}(v) + \sum_j M_j\,D\,\delta_{x_j}(v) \tag{2}$$

or equivalently (as it can be easily proved from the convexity of functional $\boldsymbol{J}$, which is the consequence of the relation $D^2\boldsymbol{J}(u;v,w) = \boldsymbol{a}(v,w)$, and $\boldsymbol{a}(v,v) \geq 0 \ \forall u,v,w \in H^2(\Omega)$) in the form of *linear* elliptic variational equation

$$\boldsymbol{a}(u,v) + \boldsymbol{s}(u,v) = \langle \mathcal{F}, v\rangle + \sum_{i,j} (F_i\,\delta_{x_i}(v) + M_j\,D\,\delta_{x_j}(v)) \quad \forall v \in H^2(\Omega) \tag{3}$$

where

$$\boldsymbol{a}(u,v) = (E\,J\,D^2\,u, D^2\,v)_{L_2(\Omega)} = \int_\Omega E\,J\,D^2 u(x)\,D^2\,v(x)\,\mathrm{d}x\ , \quad u,v \in H^2(\Omega)\ ,$$

$$\boldsymbol{s}(u,v) = (\hat{s}(u),v)_{L_2(\tilde{\Omega})} = \int_{\tilde{\Omega}} q(x)\,u(x)\,v(x)\,\mathrm{d}x\ , \quad u,v \in H^2(\Omega)\ ,$$

$$\langle \mathcal{F}, v\rangle = \int_\Omega f(x)\,v(x)\,\mathrm{d}x\ , \quad v \in H^2(\Omega)\ ,$$

and $f \in L_p(\Omega)$ is a given load, F_i, M_j are given values of generalized forces in $x_i, x_j \in \Omega$ and δ, $D\,\delta$ are Dirac distribution and its derivative in x_i, x_j, $D\,\delta(v) = -\delta(D\,v)$, $H^2(\Omega)$ is corresponding Sobolev space, see [16], [1] and others.

Remark 1 *Slightly generalized, but still linear model of subsoil can be obtained by adding of another term into the definition of response function. Thus, if we define the response function in a little more complicated form, i.e. by the relation $\hat{s}(u; D^2\,u)(x) = q(x)\,u(x) - t(x)\,D^2\,u(x)$, we receive so called two-parametrical Pasternak model of subsoil but mathematically the corresponding 'new' problem remains again linear with all the above mentioned advantages, and way of solving. Therefore, we leave out the discussion of linear Pasternak model here; non-linear generalization of Pasternak model will appear in our further coming paper.*

Once we have verified validity of the assumptions, the statements concerning the existence and uniqueness of the linear model problem solution is obvious consequence of the classical statements concerning the existence and uniqueness of *quadratic* functional minimum, or Riesz representation theorem, see [5], [7], and so on.

For the sake of brevity, we do not provide the exact and complete formulations and proofs of such statements because in the linear case it can be easy seen that it is sufficient to show the form $((u,v)) = \boldsymbol{a}(u,v) + \boldsymbol{s}(u,v)$ is equivalent to the standard scalar product on $H^2(\Omega)$. This equivalence is an easy consequence of the real physical situation, i.e. the consequence of the following assumptions for the coefficients of the given response function, shape and material property of the body. Thus we assume that for the stiffness coefficient of the subsoil material the following relation holds $q(x) \geq 0$ for a.e. $x \in \tilde{\Omega}$ and $\exists i \in I : q(x) \geq q_0 > 0$ for a.e. $x \in \tilde{\Omega}_i$, $\tilde{\Omega}_i \subset \Omega$, $\mu(\tilde{\Omega}_i) > 0$, as well as we take into account consequences of geometrical (area of cross section) and physical situation (material), i.e. we assume the relations $J(x) \geq J_0 > 0$, $E(x) \geq E_0 > 0$ for a.e. $x \in \Omega$ hold.

Remark 2 *If one considers other type of boundary conditions, elastic (Newton) support instead of free ends, for example, then it is sufficient to change just the form $\boldsymbol{s}(.,.)$ for the form $\tilde{\boldsymbol{s}}(u,v) = \boldsymbol{s}(u,v) + \boldsymbol{b}(u,v)$, where $\boldsymbol{b} = \sum_{i=0}^{1} \boldsymbol{b}^i$, $\boldsymbol{b}^i(u,v) = \sum_{j=0}^{L} k_j^i\,D^i\,u(x_j)\,D^i\,v(x_j)$ represents work of elastic deflection or slope supports in $x_j = 0, L$, $k_j^i \geq 0$ are corresponding stiffness coefficients of supports, and no statements concerning solvability of the problem will have to be changed.*

Finally, we remind the functions $\hat{s}(.)$ and $\hat{b}(.)$ defining form $\boldsymbol{s}(.,.)$ and both boundary forms $\boldsymbol{b}^i(.,.)$ representing energy of Winkler's model of subsoil and elastic supports (Newton's boundary conditions) have the same character and are linearized cases of function from Fig. 4.

Remark 3 *Let us take shortly a look at a matrix formulation of the above mentioned matters. After standard finite element discretization we are able to restate problem (1) in a finite dimension by using matrices. Let us denote $\mathbf{v} \in \mathbb{R}^N$ finite dimensional vector corresponding to the function $v_{\mathrm{h}} \in \mathcal{V}_{\mathrm{h}} \subset H^2(\Omega)$ which is the FE approximation of $v \in H^2(\Omega)$, and $\Pi(\mathbf{v})$ approximate values of the potential energy functional $\boldsymbol{J}(v)$ from (2) in a finite-dimensional space $\mathbb{R}^N$. This is determined by matrix expression*

$$\Pi(\mathbf{v}) = \frac{1}{2}\,\mathbf{v}^{\mathrm{T}}\,\mathbf{K}_{\mathrm{B}}\,\mathbf{v} + \frac{1}{2}\,\mathbf{v}^{\mathrm{T}}\,\mathbf{K}_{\mathrm{S}}\,\mathbf{v} - \mathbf{v}^{\mathrm{T}}\,\mathbf{f}\ .$$

Algebraic form of the problem (1) reads as follows

$$\Pi(\mathbf{u}) = \min_{\mathbf{v}\in\mathbb{R}^N} \Pi(\mathbf{v})\ .$$

Here we have used the following notation: $\mathbf{K}_{\mathrm{B}}$ is the stiffness matrix of the beam, $\mathbf{K}_{\mathrm{S}}$ is the stiffness matrix of the given subsoil, and $\mathbf{f}$ is vector representing given load. Considering the above mentioned expression

$$((u,v)) = \boldsymbol{a}(u,v) + \boldsymbol{s}(u,v)$$

we are competent to write in matrix formulation

$$\mathbf{K} = \mathbf{K}_{\mathrm{B}} + \mathbf{K}_{\mathrm{S}}$$

and $\mathbf{K}$ denotes the total stiffness matrix of our system 'beam + subsoil'.

Matrix $\mathbf{K}$ may contain just constant terms, which is the case of linear problem (Winkler subsoil model). Then we are able to derive necessary and sufficient condition characterizing a minimization point of the potential Π in the form

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

and we can see that we have got a system of linear algebraic equations. When we must handle with nonlinear problems, then the stiffness matrix $\mathbf{K}$ contains terms which depend on a (still unknown) solution. This leads to a system of nonlinear equations

$$\mathbf{K}(\mathbf{u})\,\mathbf{u} = \mathbf{f}\ .$$

More details on matrix formulations and methods of their solution will be given in the second part of this article.

4. Non-linear models of subsoil

An attempt to avoid all drawbacks of the linear models mentioned above, and the effort to get more physically realistic model of subsoil with possibility to distinguish material property of surrounding elastic environment in dependence on the direction of deformation leads to the definition of non-linear bilateral or unilateral subsoil of Winkler's type where response function $\hat{s}(.)$ is no more linear one but only piecewise linear, see Fig. 4 and Fig. 5 (analogous approach can be applied to the elastic supports and definitions of their response functions $\hat{b}(.)$ or to inner point obstacles).

The exact definition of the corresponding response function can be now set up from several parts: the first one represents fictitious linear bilateral 'subsoil' (firm embedding in surrounding), and the other ones represent unilateral additions or deductions of stiffness parts corresponding to added or missing 'subsoil' material in dependence on the direction of stresses and strains.

More general cases of multi-valued response functions $\hat{s}(.)$ implying formulations of the model problems in the forms of *inclusions* or *hemivariational inequalities* are introduced in [8], for example, but omitted here.

Variational formulations of the corresponding mechanical models can now be generally written in the forms of *non-linear* variational equations, resulting forms representing potential energy will not be henceforth bilinear (neither coercive for unilateral subsoil), and therefore they can not play the role of the equivalent scalar product as they did in previous Winkler linear models.

To be able to prove the statements concerning solvability of all these non-linear boundary value problems, we have to use different approach and apparatus compared with the linear problems, see [5], for example; furthermore, in semi-coercive cases we need to use special decomposition of linear spaces of kinematically admissible displacements.

Several examples illustrating such situations have been formulated and discussed in the following text.

4.1. Bilateral model of subsoil

From the mathematical point of view, the first and natural, and probably the simplest way of generalization of the linear subsoil model is the following case of non-linear response function defined by sum of linear and two unilateral parts, i.e. by the relation

$$\hat{s}(v)(x) = q_{(o)}(x)\, v(x) + q_{(+d)}(x)\, v^+(x) + q_{(-u)}(x)\, v^-(x) \quad \text{a.e. } x \in \tilde{\Omega}_i \;, \tag{4}$$

where $v^+ \equiv \max\{0, v\}$, $v^- \equiv -\min\{0, v\}$, see scheme of the corresponding potential on Fig. 4 (we assume q_1, q_2, $q_{(o)}$, $q_{(+d)}$, $q_{(-u)} \geq 0$ and define $q_1 = q_{(o)} \mp q_{(+d)}$, $q_2 = q_{(o)} \pm q_{(-u)}$, for example).

The result of usage such a function in the model definition is that the problem loses its linearity, regardless the corresponding potential has, considering problems of solvability, very pleasant property: it still remains convex, differentiable, and its gradient is monotone.

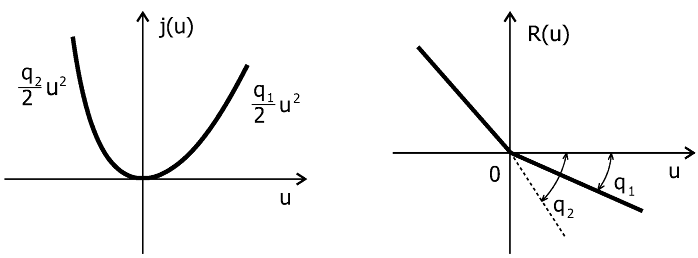

Fig.4: Scheme of non-linear subsoil model of Winkler's type: potential and response function

Mathematical formulation of the model problem with this form (4) of generalized response function $\hat{s}(.)$ is formally the same as formulation of the problem (1) (resp. (3)). The only one but still crucial difference is just in the distinct definition of the response function $\hat{s}(.)$. This new definition of the function $\hat{s}(.)$ claims quite different approach in analysis of the problem: necessity of usage of suitable statements concerning solvability of the problems with monotone operators or, in case of symmetry, usage of some of adequate theorems of variational calculus.

Thus some of convenient statements concerning the studied problem with non-linear bilateral model of subsoil and its solvability can be now founded on the following non-linear variant of Lax-Milgram Theorem (or on some suitable variant of fundamental theorem of variational calculus), see [5] for example.

Theorem 1 *Let $\boldsymbol{H}$ is Hilbert space and $\boldsymbol{a} : \boldsymbol{H} \times \boldsymbol{H} \rightarrow \boldsymbol{R}^1$ is given form such that following assumptions of monotonicity and boundness hold, i.e.: $\exists \alpha, \beta = konst. > 0$ such that*

$$\alpha \, \|v-u\|_H^2 \leq \ a(v, v-u) - a(u, v-u) \quad \forall v,u,w \in \boldsymbol{H} \ ,$$

$$|a(u,w) - a(v,w)| \leq \beta \, \|u-v\|_H \| \, w\|_H \quad \forall v,u,w \in \boldsymbol{H} \ .$$

Then $\forall \mathcal{F} \in \boldsymbol{H}^* \ \exists! \ u_{\mathcal{F}} \in \boldsymbol{H} : a(u_{\mathcal{F}}, v) = \langle \mathcal{F}, v \rangle \ \forall v \in \boldsymbol{H}$.

It can be now easily proved that under the same assumptions on material and geometrical coefficients E,J as in the above and in addition to that for $q_{(j)}(x) \geq q_j > 0$, $j = o, +d, -u$, $q_{(o)}(x) > q_{(j)}(x)$, $j = +d, -u$ for a.e. $x \in \tilde{\Omega}_i$ and the choice $\boldsymbol{H} = H^2(\Omega)$ following statement holds

Theorem 2 *Problem (1) (resp. (3)) for definition of response function $\hat{s}(.)$ by the relation (4) has unique solution.*

Remark 4 *Note, it is not necessary to choose any special types of boundary conditions to accomplish the proof of the existence and uniqueness of the solution for this non-linear model problem. It can be easily seen that the statement of the theorem 2 remains valid for all classical (linear) and non-classical (linear or non-linear) boundary conditions of monotone type (Newton non-linear conditions given through definition of $\hat{b}(.)$ illustrated on the Fig. 4, for example), and for all their combinations.*

4.2. Unilateral model of subsoil

One can easily verify, that even improved mathematical model defined in previous paragraph by means of non-linear but still bilateral (bonded) subsoil model represented by piecewise linear response function of the form (4) is not quite satisfactory one; it has no ability to describe complete or at least simplified reality in the sense of possibility of local disconnecting of the set beam and subsoil, and it has been enforcing the existence of tensile stresses upon those potentially unrestrained areas (depending on given load and boundary conditions).

That is why we have been using new approach; by the means of trivial but still quite natural manipulation of the form and value of the function representing response of subsoil, see Fig. 2, i.e. through the stiffness coefficient of the surrounding elastic environment and by using suitable limiting process $q_{(o)}, q_{(-u)} \rightarrow 0^+$ one gets at model of single (unilateral) one-sided real subsoil of Winkler's type, see scheme of its potential in the Fig. 5.

Similarly, if we assume that $q_{(o)} \equiv 0$ (unreal part of subsoil is omitted) and $q_{(+d)}, q_{(-u)} \not\equiv$ $\not\equiv 0$ (only real one-sided subsoils are taken into account), then we get at very interesting non-linear model of special environment consisting just of the two uncoupled one-sided subsoil models of previous Winkler's type, see Fig. 2d.

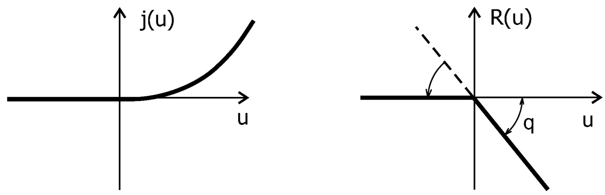

Fig.5: Scheme of unilateral subsoil model of Winkler's type : potential and response function

Analogous approach can be used likewise for definition of generalized non-linear and unilateral inner or boundary elastic supports (Newton non-linear boundary conditions, for example). In such and similar cases, the free body (beam, plate) can have let some degrees of freedom depending on the type of boundary conditions (number of d.o.f. $= 0, 1$ or 2 for beam), thus the choice of the concrete type of boundary conditions plays the crucial role in suggestion of the correct mathematical model of the pseudointeractive set of body and unilateral foundation, namely for the proof of its solvability and design of the computational methods.

Some of the difficulties concerning solution of the suggested one-sided subsoil model can be illustrated on the following simple example corresponding to the model problem with given response function of the form $\hat{s}(v)(x) = q(x)\, v^+(x)$. The resulting operator is then only monotone one but not strongly monotone one on the entire space $\boldsymbol{H}$, and corresponding stiffness matrix $\mathbf{K}$ in FE model is singular and only semidefinite one. Thus, the operator (stiffness matrix $\mathbf{K}$ in discrete model) does not fulfill necessary assumption and we can not use directly Lax-Milgram Theorem (neither in continuous nor in discrete case of the model problem) neither we can prove coerciveness of the corresponding functional of the entire potential energy (2) on the entire original space $\boldsymbol{H}$.

Analyzing the situation, we can easily verify there exist only two chances how we can get at monotony of the corresponding form (or operator) or coercivity of the corresponding potential (2); the first one is the choice of suitable type of *boundary conditions* eliminating free movement of the body (Dirichlet boundary conditions, for example) while the second one consists of enforcement of *transformation* of the semi-coercive problem to the coercive one (through *additional conditions* guarantying the existence of the solution of the problem only on a smaller set of kinematically admissible deflections, i.e. on the suitable proper subset of the original space $\mathcal{V} \subset H^2(\Omega)$).

In the first mentioned case, due to 'fixing' type of boundary conditions the problem will be coercive one and we can apply the standard approach; thus we can use Theorem 1 with new definition of the linear space of the kinematically admissible displacements $\boldsymbol{H} \equiv \mathcal{V} \subset H^2(\Omega)$, and then we can prescribe suitable formulae having the property of equivalent norm to the standard norm $\|.\|_{H^2(\Omega)}$ defined on $H^2(\Omega)$. This can be realized through a variant of Friedrichs inequality [16] and the space of kinematically admissible displacements $\mathcal{V}$ (holding stable homogeneous boundary condition), for example. Note that, in this special situation, the problems of modeling (in the sense of theoretical as well as finite-dimensional and computational aspects) are quite analogous to the coercive situation with non-linear bilateral Winkler subsoil including all difficulties arising just from the non-linearity of the problem.

In the second case which is more realistic and better describing the practical situations represented in mathematical model by free body in contact with real one-sided subsoil we suggest to use completely different and new approach. First of all, we simplify the situation by elimination of rigid displacement $\mathcal{R} \equiv \mathbf{P}^1$; note as $\mathcal{K}$ the cone of kinematically admissible rigid displacements defined by relation $\mathcal{K} = \mathcal{R} \cap \mathcal{V}$ and as $\mathcal{K}_\mathcal{V}$ its subcone $\mathcal{K}_\mathcal{V} \subset \mathcal{K}$ where $\mathcal{K}_\mathcal{V} = \{v \in \mathcal{K} \mid \boldsymbol{a}(v,v) + \boldsymbol{s}(v,v) = 0\}$. Then, the solution of semi-coercive problem can be found on the negative polar cone $\mathcal{K}_\mathcal{V}^\ominus$ to the cone $\mathcal{K}_\mathcal{V}$ where the operator corresponding to the problem behaves like the coercive one.

The concrete form of the cone $\mathcal{K}_\mathcal{V}$ depends on the given linear space $\mathcal{V}$, i.e. on the type of given boundary conditions as well as on the definition of the form $\tilde{\boldsymbol{s}}(.,.)$, i.e. on the model subsoil design. For the sake of simplicity, we are going to illustrate suggested way and method of the proof of the existence of solution for the given mathematical model problem only through a simple elementary example with one degree of freedom: number of d.o.f. = 1, thus we define $\mathcal{V} = \{v \in H^2(\Omega) \mid D\,v(0) = 0\}$ (for different model given by circle plate and Sobolev weighted space $\mathcal{V}$, see [14]).

Fundamental tool for our approach is the following best approximation theorem. This statement gives us possibility of useful and unique decomposition of the space $\mathcal{V}$ into the orthogonal sum of the cone $\mathcal{K}_\mathcal{V}$ and its polar cone $\mathcal{K}_\mathcal{V}^\ominus$, i.e. into the form $\mathcal{V} = \mathcal{K}_\mathcal{V} \oplus \mathcal{K}_\mathcal{V}^\ominus$. Thus we remind following general theorems, see [2], for example.

Theorem 3 *Let $\boldsymbol{H}$ be Hilbert space and $\Lambda \subset \boldsymbol{H}$ convex closed subset. Then* $\forall \boldsymbol{x} \in \boldsymbol{H}$ $\exists!\ \boldsymbol{y} \in \Lambda :\ \|\boldsymbol{x}-\boldsymbol{y}\|_H \le \|\boldsymbol{x}-\boldsymbol{z}\|_H\ \forall \boldsymbol{z} \in \Lambda$.

The element $\boldsymbol{y} \in \Lambda$ is equivalently characterized by the inequality

$$(\boldsymbol{x}-\boldsymbol{y}, \boldsymbol{z}-\boldsymbol{y})_H \le 0 \quad \forall \boldsymbol{z} \in \Lambda\ .$$

The element $\boldsymbol{y} = P_\Lambda(\boldsymbol{x})$ and operator $P_\Lambda : \boldsymbol{H} \to \Lambda$ is said to be best approximation of $\boldsymbol{x}$ on Λ and projection of the best approximation on Λ.

By the means of the best approximation operator defined in the previous theorem, we can easily prove the statement concerning the unique orthogonal decomposition of Hilbert space $\boldsymbol{H}$ into two cones, i.e. following theorem, for details see again [2].

Theorem 4 *Let $\boldsymbol{H}$ be Hilbert space and $\Lambda \subset \boldsymbol{H}$ given convex closed cone with vertex in* $\mathbf{0}$, *i.e.* $\mathbf{0} \in \Lambda$.

Then $\forall \boldsymbol{x} \in \boldsymbol{H}$ $\exists!\ \{\boldsymbol{y},\boldsymbol{z}\} \in \Lambda \times \Lambda^\ominus :\ \boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}$, *where* $(\boldsymbol{y},\boldsymbol{z})_H = 0$.

Furthermore, mappings P_Λ, $P_{\Lambda^\ominus}$ and elements $\boldsymbol{y} = P_\Lambda(\boldsymbol{x})$, $\boldsymbol{z} = P_{\Lambda^\ominus}(\boldsymbol{x})$ are the best approximation operators, and orthogonal projections of $\boldsymbol{x} \in \boldsymbol{H}$ on cone Λ and its negative polar cone $\Lambda^\ominus$.

Thus, we have prepared everything (apparatus, auxiliary statements, ideas, and so on) to be able to formulate the necessary and sufficient condition of solvability and to illustrate the main idea and way of the proof of the statement concerning existence and uniqueness of the solution of the model problem representing free body lying on the elastic one-sided subsoil.

Theorem 5 *Let E,J are given material and geometrical coefficients, and q is given coefficient of subsoil stiffness of the model problem, and let relations $E(x) \geq E_0 > 0$, $J(x) \geq J_0 > 0$ for a.a. $x \in \Omega$ and $q(x) \geq q_0 > 0$ for a.a. $x \in \tilde{\Omega}_i$ hold, where measure $\mu(\tilde{\Omega}_i) > 0$.*

Then the necessary and sufficient condition guarantying the existence of the model problem solution, i.e. solvability of the problem (1) (resp. problem (3)) on $\mathcal{V}$ and with the function $\hat{s}(.)$ defined by the relation $\hat{s}(v)(x) = q(x)\, v^+(x)$ has the form

$$\langle \mathcal{F}, 1 \rangle + \sum_i F_i \geq 0 \tag{5}$$

(or in detail $\int_\Omega f(x)\, dx + \sum_i F_i \ \geq 0$).

If the following relation holds (existence and uniqueness condition)

$$\langle \mathcal{F}, 1 \rangle + \sum_i F_i > 0 \ , \tag{6}$$

then suggested mathematical model of the problem with elastic one-sided subsoil is correct one, i.e. problem (1) (resp. problem (3)) with function $\hat{s}(.)$ defined by $\hat{s}(v)(x) = q(x)\, v^+(x)$ has exactly one solution.

Proof of the necessity of the condition (5) from the first statement of Theorem 5 is easy one. It is enough to formulate well known Euler necessary and sufficient condition characterizing extreme of convex functional (2), and then use as a test function any rigid deflection $v \in \mathcal{K}_\mathcal{V}$ (corresponding cone of kinematically admissible rigid displacements for the studied model problem is defined by the relation $\mathcal{K} = \mathcal{R} \cap \mathcal{V} \equiv \mathbf{P}_0$, thus we have $\mathcal{K}_\mathcal{V} \equiv (\mathbf{R}^1)^-$ and corresponding rigid deflection has the form $v(x) = c$, $c \in \mathbf{R}$, $c \leq 0$).

Then, by using of the property of the form $\boldsymbol{s}(.,.)$ with response function defined by $\hat{s}(v)(x) = q(x)\, v^+(x)$ we arrive to the condition (5).

Proof of the sufficiency of the condition (5) from the first statement of Theorem 5 requires to use orthogonal decomposition of the linear space $\mathcal{V}$ in the sense of the Theorem 4. Note, due to character of the chosen model example, we can restrict ourselves only to the symmetry case of the form $\boldsymbol{a}$, i.e. to the variational formulation.

Firstly, we define negative polar cone $\mathcal{K}_\mathcal{V}^\ominus$ to the cone $\mathcal{K}_\mathcal{V}$ by the relation $\mathcal{K}_\mathcal{V}^\ominus = \{v \in \mathcal{V} \mid (v, w)_H \leq 0 \ \forall w \in \mathcal{K}_\mathcal{V}\}$, thus, due to Theorem 4, the decomposition $\mathcal{V} = \mathcal{K}_\mathcal{V} \oplus \mathcal{K}_\mathcal{V}^\ominus$ holds. Next, considering character of the functional (2) we see it is sufficient to prove its coercivity only on $\mathcal{K}_\mathcal{V}^\ominus$, rest of the proof is simple consequence of the well known theorems of variational calculus, see [5] for example.

Then, through the definition of standard scalar product on $H^2(\Omega)$, see [4] or [16] for example, we can easily prove the cone $\mathcal{K}_\mathcal{V}^\ominus$ is characterized just by those functions $v \in \mathcal{V}$ which fulfill the condition $\int_\Omega v(x)\, \mathrm{d}x \geq 0$.

Further, for any deflection function $v \in \mathcal{V}$ holds the decomposition $v = p \oplus \bar{v}$, where $p \in \mathcal{K}_\mathcal{V}$ and $\bar{v} \in \mathcal{K}_\mathcal{V}^\ominus$, i.e. $p(x) = c$, $c \in \mathbf{R}$, $c \leq 0$ and $\int_\Omega \bar{v}(x)\, \mathrm{d}x \geq 0$.

Finally, characterization of the cones $\mathcal{K}_\mathcal{V}$ and $\mathcal{K}_\mathcal{V}^\ominus$, together with the orthogonality condition imply following, for the proof crucial property of the decomposition

$$\text{i)} \qquad c \leq 0 \ ; \qquad \left(c < 0 \ \Rightarrow \ \int_\Omega v(x)\, \mathrm{d}x = 0 \ \equiv \ (1, v)_0 = 0 \right) \ , \tag{7}$$

$$\text{ii)} \qquad \int_{\Omega} v(x)\,\mathrm{d}x \geq 0 \; ; \qquad \left(\int_{\Omega} v(x)\,\mathrm{d}x > 0 \;\Rightarrow\; c = 0 \right) \; . \tag{8}$$

Thus, and by means of the following estimation (forces F_i, M_j are neglected for a while)

$$\boldsymbol{J}(v) \geq C_1\, |\bar{v}|^2_{2,2,\Omega} + C_2 \int_{\Omega_i} \left((p+\bar{v})^+ \right)^2 - p \int_{\Omega} f(x)\,\mathrm{d}x - \int_{\Omega} f(x)\,\bar{v}\,\mathrm{d}x$$

we receive with help of Poincaré inequality, see [4], [16], coerciveness of the functional $\boldsymbol{J}$, i.e. the existence of a minimum of $\boldsymbol{J}$ on the $\mathcal{K}^{\ominus}_{\mathcal{V}}$, i.e. the existence of the model problem solution is proved.

Uniqueness of the problem solution is then consequence of the strict convexity of the problem potential $\boldsymbol{J}$ on the cone $\mathcal{K}^{\ominus}_{\mathcal{V}}$ which is implied by the condition (6).

4.3. Model of unilateral rigid subsoil

Finally, we only briefly mention the last theoretical but from the practical point of view not very important case, i.e. the other limit case in our demarcated class of subsoil models defined now by the perfect rigid subsoil in the depth $\Delta \geq 0$ from the beam, see Fig. 2f (the case of $\Delta = 0$). This restriction to the perfectly rigid subsoil implies substitution of the linear space $\mathcal{V}$ by the corresponding convex set of constraints

$$\mathcal{M} = \{ v \in \mathcal{V} \mid v(x) \geq \Delta,\ \text{a.a. } x \in \Omega,\ \Delta \in \mathbf{R}^1,\ \Delta \geq 0 \}$$

and by omitting the form $\boldsymbol{s}(.,.)$ representing elastic subsoil in the definition of the functional $\boldsymbol{J}$. Thus the effects of perfectly rigid subsoil are included in the mathematical model only by means of definition of the set $\mathcal{M}$. Corresponding mathematical formulation can have then the form of the minimization problem (1) but over the convex set $\mathcal{M}$ instead of the linear space $\mathcal{V}$ or equivalently form of the inequality of the 1$^{\text{st}}$ kind, see [6] for example, i.e. following form

$$u \in \mathcal{M} : \; \boldsymbol{a}(u, v-u) \geq \langle \mathcal{F}, v-u \rangle \; + \\ + \sum_i F_i\, \delta_{x_i}\,(v-u) + \sum_j M_j\, \delta_{x_j}\,(D\,v - D\,u) \quad \forall v \in \mathcal{M} \; . \tag{9}$$

From the mathematical point of view, solvability problem of the minimization problem (1) with modified functional $\boldsymbol{J}$ over constraint set $\mathcal{M}$, or equivalently solvability problem of the corresponding variational inequality (9) is quite analogous to the situation analyzed in previous paragraph (now with modified Theorem 1, i.e. with help of Lions-Stampacchia Theorem, see [6] for example). Note, we have to use different approach to prove solvability of the model problem (9) in this case. It is possible to use non-orthogonal decomposition of the linear space $\mathcal{V}$, see [7], [12], for example.

4.4. Generalization of subsoil model

All introduced subsoil models within the frame of defined class can be generalized by many natural ways, thus the class of subsoil models can be easy enlarged to cover all standard and practical applications. For example, one can define the response function as a continuous piecewise linear function with given restriction Δ, see Fig. 6, where $\Delta = \sum_i L_i$,

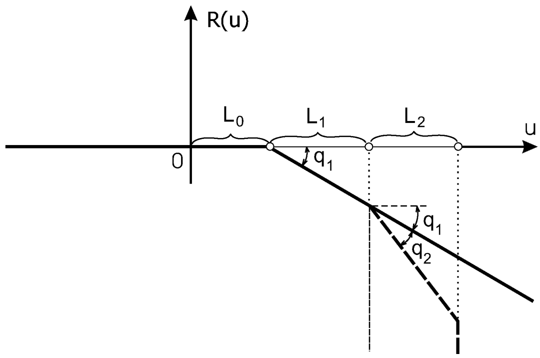

Fig.6: Graph of function representing subsoil model

corresponding mathematical model can be obtained by combination of the problem (9) with adding on the corresponding function definition $\hat{s}(.)$ and form s(.,.). Furthermore, one can also use general non-linear monotone continuous function and get the analogous results.

Similarly, one can use all generalized definitions of subsoil models in previous as models for definition of response functions of unilateral elastic supports or inner point obstacles, i.e. for definition of the functions $\hat{b}(.)$ and forms $\boldsymbol{b}^i(.,.)$.

Finally, essential generalization of the mathematical model representing even possible failure or damage of the material of supports, obstacles or any part of subsoil can be obtained by means of using multi-value function $\hat{s}(.)$ in model problem definition; formulation of the problem leads then to the inclusion problem form, or equivalently to the form of hemivariational inequality, see [8] for details and theory, or see [10], [13] for application to the problem of coupled thermoelasticity.

5. Conclusion

Throughout the paper we have considered special class of one–parametrical subsoil models of Winkler's type represented by piecewise linear response functions. Methods for analysis of the solvability for model class problems have been presented, conditions of solvability in semi-coercive cases have been given. Thus, all introduced non-linear mathematical models are correct ones, the statements concerning the existence and uniqueness (under additional conditions) of their solutions have been formulated, and their proofs have been discussed.

Next, in the second part of our paper *Part II. Computational Aspects and Applications*, the FE approximation and corresponding discrete model as well as numerical methods used for building of the computational model and some illustrative examples and applications will be given.

Finally, in similar way, we can take into account 'direct' generalization of Winkler's models, i.e. two-parametrical subsoil models of Pasternak's type. Parameters in Pasternak's model are not coupled ones, thus they can be generalized independently, and standard Pasternak response function $\hat{s}(u; D^2\, u(x)) = q(x)\, u(x) - t(x)\, D^2\, u(x)$ can be analogically rewrite in the following generalized form

$$\hat{s}^{(i,\,j)}(u^{(i)}; (D^2\, u)^{(j)}(x) \equiv q(x)\, u^{(i)}(x) - t(x)\, (D^2\, u(x))^{(i)}\;, \quad i, j = 0, 1, +\,, -\;,$$

where the notation $f^{(0)} \equiv 0$ and $f^{(1)} \equiv f$ have been used. Regarding another way of generalization, see [8] for example. Thus, corresponding mathematical models (generalized

Pasternak) can be analyzed by using analogical approach as mentioned above and all presented methods can be applied to obtain completely analogous results. The gentle reader will be able to familiarize with some variants of these models and analogous class of non-linear response functions in our following papers.

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