APPLICATION OF THE DIFFERENTIAL QUADRATURE METHOD TO THE LONGITUDINAL VIBRATION OF NON-UNIFORM RODS

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Differential Quadrature Method (DQM) has a very wide applications in the field of structural vibration of various elements such as beams, plates, cylindrical shells and tanks. One of the most advantages of the DQM is its simple forms for nonlinear formulations using the Hadamard product. In this paper the free vibration of a general non-uniform rod were studied. The non-dimensional natural frequency and the normalized mode shapes of the non-uniform rod of free and clamped boundary conditions were obtained by using 15 point DQ method and compared to those of references [8] and [9]. Results shows good agreement with the previous analytical solutions. The effect of the varying cross section area on the vibration were studied. This work reflects the power of the DQM in solving non-uniform problems.

Key words: *non-uniform rod, vibration, differential quadrature method*

1. Introduction

The method of differential quadrature discretizes any derivative at a point by a weighted linear sum of functional values at its neighbouring points.The basic procedure in the DQM is the determination of weighting coefficients. Based on the idea of integral quadrature the differential quadrature method was first introduced by Richard Bellman 1972 [1]. Bellman used two procedures to obtain the weighting coefficients. The first procedure used a simple functions as test functions but unfortunately when the sampling points are relatively large (say 13) the coefficient matrix become ill conditioned. The second procedure is similar to the first one with the exception that the coordinates of grid points should be chosen as the roots of the Nth order Legendre polynomial. Most early researches was based on the first procedure because the grid points can be chosen arbitrarily. The details of the DQ method can be found in reference [2].

1.1. Treatment of boundary conditions

All the boundary value problems contains a number of boundary conditions which must be satisfied. When there is one boundary condition at each boundary point there is no problems, but if there is more than one boundary condition at each boundary point, four counted methods for the treatment of boundary conditions in the DQM. The first method is the so called δ-technique proposed by Bert et al. [3], which restricts the boundary condition at boundary points and derivative boundary conditions at the δ points, which have very small distance δ away from the boundary. And so this approach can not satisfy derivative

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boundary conditions exactly at boundary points. The solution accuracy depends on the proper choice of δ. If the value of δ is small enough, the approach produce good results in some situations such as clamped condition, however, failed to work well in the other situations such as simply-supported and free edges. On the other hand, too small values for δ will cause ill-conditioned matrices.

The second method which introduced by Wang and Bert [4] can be referred as Modifying Weighing Coefficient Matrices (MWCM). This method depends on imposing the derivative boundary conditions into the weighing coefficient matrices. MWCM gives the most accurate results in treating simply supported boundary conditions but it fails to treat other boundary conditions such as clamped and free.

The third method introduced by Shu and Du [5] can be referred as Substituting Boundary conditions into Governing Equations (SBCGE). This method substitutes the boundary conditions directly into the governing equation and the derivative boundary conditions can be coupled to provide the values of the function values at to points optimally adjacent to the first and last points. This method works well at any type of boundary conditions, and so this method treats the drawbacks of the previous methods. Although SBCGE is an efficient method in treating boundary conditions, it suffers some lack of generality and consumes much time and effort in preparing the modified matrices.

The fourth method introduced by Shu and Du [6] can be referred as Coupling Boundary Conditions with Governing Equations (CBCGE). By this method the vibration problem can be transformed into a general boundary value problem. This method works efficiently with any combinations of boundary conditions, but it gives less accurate results than those obtained by (SBCGE).

2. Differential quadrature method

The main idea of the DQM is that the derivative of a function at a sample point can be approximated as a weighted linear summation of the value of the function at all of the sample points in the domain. Using this approximation, the differential equation is reduced to a set of algebraic equations. The number of equations depends on the selected number of sample points.

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \cong \left[C_{ij}^m\right] \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} , \quad i,j = 1,2,\ldots,N , \tag{1}$$

where $f(x_i)$ is the value of the function at the sample point x_i and C_{ij}^m are the weighting coefficients of the m^{th}-order differentiation attached to these functional values. Quan et al. [7] introduced a Lagrangian interpolation polynomial to overcome the numerical ill-conditions in determining the weighting coefficients C_{ij}^m

$$f(x) = \sum_{i=1}^{N} \frac{M(x)}{(x - x_i)\, M_1(x_i)}\, f(x_i) , \tag{2}$$

where

$$M(x) = \prod_{j=1}^{N} (x - x_j) ,$$

$$M_1(x_i) = \prod_{j=1}^{N} (x_i - x_j) \quad \text{for} \quad i = 1, 2, \ldots, N \; .$$

Substituting equation (2) into equation (1) yields

$$C^1_{ij} = \frac{M_1(x_i)}{(x_j - x_i)\, M_1(x_j)} \quad \text{for} \quad i, j = 1, 2, \ldots, N \quad \text{and} \quad i \neq j \; . \tag{3}$$

$$C^1_{ii} = - \sum_{j=1,\, j\neq i}^{N} C^1_{ij} \quad \text{for} \quad i = 1{,}2, \ldots, N \; . \tag{4}$$

After the sample points have been selected, the coefficients of the first order weighting matrix can be obtained from equations (3) and (4). The number of the test functions must exceed the highest order of the derivative in the governing equations; that is $N > m$. Higher order coefficient matrices can be obtained from the first order weighting matrix as follows

$$C^2_{ij} = \sum_{k=1}^{N} C^1_{ik}\, C^1_{kj} \quad \text{for} \quad i, j = 1, 2, \ldots, N \; , \tag{5}$$

$$C^3_{ij} = \sum_{k=1}^{N} C^1_{ik}\, C^2_{kj} \quad \text{for} \quad i, j = 1, 2, \ldots, N \; , \tag{6}$$

$$C^4_{ij} = \sum_{k=1}^{N} C^1_{ik}\, C^3_{kj} \quad \text{for} \quad i, j = 1, 2, \ldots, N \; . \tag{7}$$

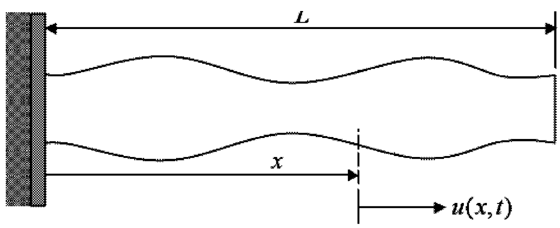

Fig.1: Configuration of a non-uniform axially vibrating rod

3. Formulation of the problem of vibration of a non-uniform rod

Non-uniform rods are rods with varying cross section. The longitudinal vibration of a non-uniform rod is governed by the differential equation [8]

$$\frac{\partial}{\partial \bar{x}} \left[E\, A(\bar{x})\, \frac{\partial u}{\partial \bar{x}} \right] = \varrho\, A(\bar{x})\, \frac{\partial^2 u}{\partial t^2} \; . \tag{9}$$

Assume that the solution of equation (9) to be in the form

$$u(\bar{x}, t) = W(\bar{x})\, \mathrm{e}^{\mathrm{i}\, \bar{\omega}\, t} \; . \tag{10}$$

Substituting equation (10) into equation (9) we get

$$E\,A(\bar{x})\,\frac{\mathrm{d}^2W}{\mathrm{d}\bar{x}^2} + E\,\frac{\mathrm{d}A(\bar{x})}{\mathrm{d}\bar{x}}\,\frac{\mathrm{d}W}{\mathrm{d}\bar{x}} + \varrho\,A(\bar{x})\,\bar{\omega}^2\,W = 0\ . \tag{11}$$

Equation (11) can be transformed to a non-dimensional form as follows,

$$\frac{\mathrm{d}^2W}{\mathrm{d}x^2} + \frac{1}{S(x)}\,\frac{\mathrm{d}S(x)}{\mathrm{d}x}\,\frac{\mathrm{d}W}{\mathrm{d}x} + \Omega^2\,W = 0 \tag{12}$$

with the non-dimensional coefficients

$$x = \frac{\bar{x}}{L}\ , \qquad S(x) = \frac{E\,A(\bar{x})}{E\,A_0}\ , \qquad \Omega^2 = \frac{\varrho\,\bar{\omega}^2\,L^2}{E}\ ,$$

where L is the length of the rod, E is the modulus of elasticity, A_0 is the rod cross section at the position $x = 0$, ϱ is the mass density and Ω is the non-dimensional frequency. It should be noted that if $S(x) = 1$, equation (12) reduces to the equation of the uniform rod.

Applying the Differential Quadrature discretization to the non-dimensional governing equation (12)

$$\sum_{j=1}^{N} B_{ij}\,W_j + \frac{1}{S(x)}\,\frac{\mathrm{d}S(x_i)}{\mathrm{d}x}\sum_{j=1}^{N} A_{ij}\,W_j + \Omega^2\,W_i = 0\ , \tag{13}$$

where A_{ij} and B_{ij} are the weighting coefficients matrices of the first and second order respectively.

To complete the formulation we have to discuss the boundary conditions. The solution will be obtained for two types of boundary conditions which are clamped-clamped and clamped-free. For clamped-clamped (C-C) we have

$$W(0) = W(1) = 0\ , \tag{14}$$

for camped-free (C-F)

$$W(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}W(1)}{\mathrm{d}x} = 0\ . \tag{15}$$

The boundary conditions equations (14) and (15) can be represented in the Differential Quadrature form as

$$W_1 = W_N = 0\ , \tag{16}$$

$$W_1 = 0 \quad \text{and} \quad \sum_{j=1}^{N} A_{Nj}\,W_j = 0\ . \tag{17}$$

For the two sets of boundary conditions we can use the MWCM approach developed by Wang and Bert [5] which modified the weighting coefficient matrices to include the boundary conditions.

By this procedure equation (13) and equation (16) can be written in the form

$$\sum_{j=2}^{N-1} \bar{B}_{ij}^{(\mathrm{CC})}\,W_j + \frac{1}{S(x_i)}\,\frac{\mathrm{d}S(x_i)}{\mathrm{d}x}\sum_{j=2}^{N-1} \bar{A}_{ij}^{(\mathrm{CC})}\,W_j + \Omega^2\,W_i = 0 \tag{18}$$

and equation (13) and equation (17) can be written in the form

$$\sum_{j=2}^{N-1} \bar{B}_{ij}^{(\mathrm{CF})}\, W_j + \frac{1}{S(x_i)}\, \frac{\mathrm{d}S(x_i)}{\mathrm{d}x} \sum_{j=2}^{N-1} \bar{A}_{ij}^{(\mathrm{CF})}\, W_j + \Omega^2\, W_i = 0\ , \tag{19}$$

where $\bar{A}_{ij}^{(\mathrm{CC})}$ and $\bar{B}_{ij}^{(\mathrm{CC})}$ are the first and second order modified weighting coefficient matrices containing the clamped-clamped boundary conditions and $\bar{A}_{ij}^{(\mathrm{CF})}$ and $\bar{B}_{ij}^{(\mathrm{CF})}$ are those containing the clamped-free boundary conditions respectively.

Equation (18) and (19) can be written in the following compact form

$$\left(\bar{\mathbf{B}}^{(\mathrm{CC})}\, \vec{\mathbf{W}}\right) + \left(\mathbf{S} \circ \bar{\mathbf{A}}^{(\mathrm{CC})}\right) \vec{\mathbf{W}} + \Omega^2\, \vec{\mathbf{W}} = 0\ , \tag{20}$$

$$\left(\bar{\mathbf{B}}^{(\mathrm{CF})}\, \vec{\mathbf{W}}\right) + \left(\mathbf{S} \circ \bar{\mathbf{A}}^{(\mathrm{CF})}\right) \vec{\mathbf{W}} + \Omega^2\, \vec{\mathbf{W}} = 0\ , \tag{21}$$

where $\circ$ denotes the Hadamard product and $\mathbf{S}$ is an $N \times N$ matrix whose columns are identical and each column consists of the values of the term $(1/S(x_i))\, \mathrm{d}S(x_i)/\mathrm{d}x$ at each discrete point.

Equations (20) or (21) can be reduced to a set of linear equations, eigenvalue problem, which can be solved using standard eigen solver.

It should be noted that the modified matrices are completely different from the original weighting coefficient matrices. For example to include the C-C boundary conditions the first order coefficient must be in the form

$$\begin{gathered}
\bar{A}_{ij}^{(\mathrm{CC})} = \begin{bmatrix} 0 & A_{1,2} & \cdots & A_{1,N-1} & 0 \\ 0 & A_{2,2} & \cdots & A_{2,N-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & A_{N,2} & \cdots & A_{N,N-1} & 0 \end{bmatrix}\ , \\
\bar{A}_{ij}^{(\mathrm{CF})} = \begin{bmatrix} 0 & A_{1,2} & \cdots & A_{1,N-1} & A_{1,N} \\ 0 & A_{2,2} & \cdots & A_{2,N-1} & A_{2,N} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & A_{N-1,2} & \cdots & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\ ,
\end{gathered} \tag{22}$$

and $\bar{B}_{ij}^{(\mathrm{CC})}$ can be obtained simply by the relation

$$\bar{B}_{ij}^{(\mathrm{CC})} = A_{ij}\, \bar{A}_{ij}^{(\mathrm{CC})}\ . \tag{23}$$

For more details see [5].

4. Results and discussion

4.1. Choice of Sampling Points

Choosing the number and type of sampling points has a great effect on the accuracy of the DQM results. Equal spacing sampling points used in earlier papers gives some inaccurate results. It is found that the optimal selection of the sampling points in the vibration problems is the normalized Chebyshev-Gauss-Lobatto points,

$$x_i = \frac{1}{2} \left[1 - \cos\left(\frac{i-1}{N-1}\, \pi\right)\right]\ , \quad i = 1, 2, \ldots, N\ . \tag{24}$$

But there are various types of grid distribution which gives an acceptable results, for more details see ref. [2].

In order to study our results, calculations by using 15 points DQ have been done for both uniform C-C, C-F rod ($S(x) = 1$) and a non-uniform C-C, C-F rod under three sets of area variations. The exact natural frequencies of these cases are introduced as found in paper [8]. The relative difference typed in tables (1) and (2) is to represent the accuracy of the DQ method and this relative difference is equal to (Present-Reference)/Reference. Examining tables (1) and (2) it is quite clear that the results are in excellent agreement with the previous data.

To complete the picture of the longitudinally vibrating non-uniform rod, the first four normalized mode shapes are plotted for the two set of area variations and boundary conditions. It should be noticed that the normalized mode shapes of the non-uniform rod suffers the evanescent behaviour which is a common feature of the mode shapes of the non-uniform rods and beams.

Rod type	Ω_1	Ω_2	Ω_3	Ω_4
Uniform [present]	1.5708	4.7124	7.8540	10.9956
Non-uniform [9]	1.1656	4.6042	7.7899	10.9499
Non-uniform [present]	1.1656	4.6042	7.7899	10.9499
Relative difference %	0.0000	0.0000	0.0000	0.0000

Tab.1: Non-dimensional frequency for a non uniform C-F rod whose area variation is of the form $S(x) = (x + 1)^2$

Rod type	Ω_1	Ω_2	Ω_3	Ω_4
Uniform [present]	3.1416	6.2832	9.4248	12.5664
Non-uniform [8]	2.9782	6.2031	9.3716	12.5265
Non-uniform [present]	2.9782	6.2031	9.3716	12.5265
Relative difference %	0.0000	0.0000	0.0000	0.0000

Tab.2: Non-dimensional frequency for a non uniform C-C rod whose area variation is of the form $S(x) = \sin^2(x + 1)$

Tables 3 and 4 demonstrates the previously discussed boundary conditions and its effect on the natural frequency and compares these frequencies to those obtained from a uniform rod. Table 3 demonstrates the accuracy of the C-F rod with area variation $S(x) = (x + 1)^4$ and show that the method gives an acceptable results. Table 4 shows that the DQ method gives an exact results for the non-uniform rod with C-F ends with area variation $S(x) = (x + 1)^2$.

Rod type	Ω_1	Ω_2	Ω_3	Ω_4
Uniform [present]	1.5708	4.7124	7.8540	10.9956
Non-uniform [8]	—	4.4875	7.7217	10.9016
Non-uniform [present]	0.8250	4.6004	7.8910	10.9497
Relative difference %	—	2.5159	2.1925	0.4412

Tab.3: Non-dimensional frequency for a non uniform C-F rod whose area variation is of the form $S(x) = (x + 1)^4$

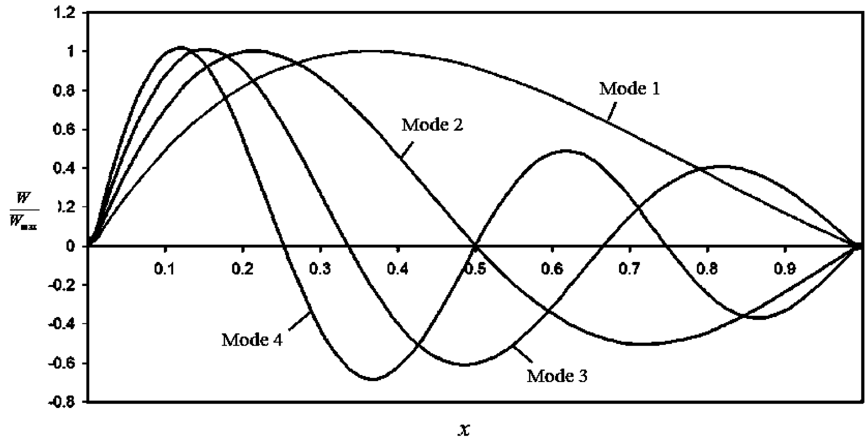

Fig.2: First four mode shapes for a non-uniform C-C rod whose area variation is of the form $S(x) = (x+1)^4$

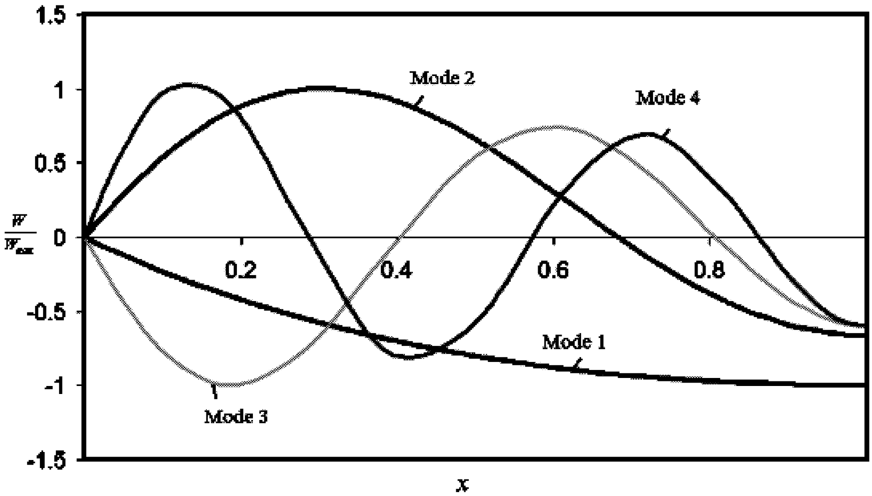

Fig.3: First four mode shapes for a non-uniform C-F rod whose area variation is of the form $S(x) = (x+1)^2$

Rod type	Ω_1	Ω_2	Ω_3	Ω_4
Uniform [present]	1.5708	4.7124	7.8540	10.9956
Non-uniform [8]	1.5176	4.7021	7.8483	10.9916
Non-uniform [present]	1.5176	4.7021	7.8483	10.9916
Relative difference %	0.0000	0.0000	0.0000	0.0000

Tab.4: Non-dimensional frequency for a non uniform C-F rod whose area variation is of the form $S(x) = \sin^2(x+1)$

5. Conclusion

From the above discussion it can be demonstrated that the DQ method is an efficient method in solving the vibration of non-uniform elements such as beams and rods. The effort in solving this problem in [8] or [9] is much greater than the computational effort by the present method. Very good accuracy is obtained by a very few grid points (15 points in our case). The power of the DQ method is its ease in treating the various combinations of boundary conditions.

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